THE NUMBER OF MEASURES ON VERY LARGE MEASURABLE CARDINALS

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Abstract. We study the possible number of normal measures on a measurable cardinal in settings where inner model techniques are unavailable. Instead, we exploit consequences of the Ultrapower Axiom to obtain our theorems. We show that the classical Kimchi–Magidor result—that the first n measurable cardinals can be strongly compact—can be combined with an arbitrary prescribed pattern for the number of normal measures they carry. We also prove that the first measurable cardinal above a supercompact cardinal can carry any given number of normal measures; the same conclusion is established for the first measurable limit of supercompact cardinals. As further applications of our techniques, we strengthen an unpublished theorem of Goldberg–Woodin and a theorem of Goldberg, Osinski, and Poveda. Our analysis circumvents both the reliance of Friedman–Magidor [14] on core model methods and the limitations of the Prikry-type forcing iterations of [17].

1. Introduction

The present manuscript studies the number of *normal measures* on a *measurable cardinal*. Measurable cardinals trace their origin to the classical work of S. Ulam [40] in connection with the Lebesgue measure problem. A cardinal κ is called *measurable* if there exists a non-principal, κ-complete ultrafilter on κ.[1] Normal measures on κ are such ultrafilters that are additionally closed under diagonal intersections. It is well-known that every measurable cardinal carries at least one normal measure. The question of how many normal measures a measurable cardinal may carry has been a significant and well-studied problem in recent decades, culminating in its complete solution by Friedman and Magidor in [14] (see the discussion below).

In this paper, we are interested in measurable cardinals that coexist with *very large cardinals*, such as strongly compact and supercompact cardinals. A cardinal κ is *strongly compact* if, for every cardinal $\lambda \geq \kappa$, there exists a fine, κ-complete ultrafilter on $\mathcal{P}_\kappa(\lambda) = \{X \subseteq \lambda : |X| < \kappa\}$. A cardinal κ is *supercompact* if, for every $\lambda \geq \kappa$, there exists a fine, normal ultrafilter on $\mathcal{P}_\kappa(\lambda)$. Both are very strong forms of measurability and strictly exceed measurable cardinals in consistency strength.

Studying the higher levels of the large cardinal hierarchy can be challenging due to the current lack of a *canonical inner model* for a supercompact cardinal. The construction of canonical inner models in set theory originated with Gödel's discovery of the universe of constructible sets, denoted by L. However, L itself cannot accommodate most large cardinals—for instance, by a classical theorem of Scott, there are no measurable cardinals in L. Nevertheless, over the years a sequence of L-like models accommodating progressively stronger large cardinals has been developed, and the inner model program aims to construct such models for all large cardinal axioms.

The inner model theoretic approach provides powerful tools and techniques for addressing fundamental questions about large cardinals. The possible number of normal measures on a measurable cardinal is a concrete example of such a question. Indeed, Kunen constructed the canonical inner model for a measurable cardinal, $L[U]$, and showed that it carries a unique normal measure. Later, Mitchell developed inner models of the form $L[\vec{U}]$, constructed from coherent sequences of measures

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[1]Such an ultrafilter U trivially induces a nontrivial, two-valued, κ-additive measure on $\mathcal{P}(\kappa)$. Indeed, the measure $\mu\colon \mathcal{P}(\kappa) \to 0, 1$ associated with U assigns the value 1 to a set if and only if it belongs to U.

$\vec{U}$, which can exhibit measurable cardinals carrying any reasonable number of normal measures. However, Mitchell's construction requires assumptions beyond the existence of a single measurable cardinal. It was not clear whether the existence of a single measurable cardinal suffices to establish the consistency of a measurable cardinal κ carrying exactly λ normal measures, for a prescribed cardinal $1 < \lambda < \kappa^{++}$.[2]

The first major step towards resolving this problem was made in the early 70's by Kunen–Paris [31], who showed that starting from a single measurable cardinal κ, one can produce a forcing extension in which the number of normal measures on κ is any cardinal λ of cofinality greater than κ^+. In particular, one can achieve the maximal possible number of normal measures, 2^{2^κ}. More recently, Apter–Cummings–Hamkins [5] addressed the first case beyond the scope of the Kunen–Paris technique; namely, starting from a single measurable cardinal κ, the authors constructed a generic extension where κ carries exactly κ^+ normal measures. Still, this left open the question of whether, from a single measurable cardinal, one can produce a model of ZFC with exactly λ normal measures for any cardinal $\lambda \leq \kappa^+$.

The above problem was completely resolved by Friedman and Magidor in their pioneering work [14]. Friedman–Magidor's work not only settled a long-standing open problem, but also skillfully exploited a number of sophisticated techniques that nowadays play a pivotal role in the field. These include forcing over canonical inner models, the use of nonstationary support iterations, generalized Sacks forcing, and self-coding forcing. The lasting influence of Friedman–Magidor's ideas in the area can be better appreciated through subsequent works like [8, 9, 11, 12, 3, 23, 4, 17, 10, 24].

One of the distinctive aspects of the Friedman–Magidor proof is that it requires forcing over a canonical inner model (see [38]). This feature is essential to their strategy for controlling the normal measures that appear in the generic extension in two ways. First, the new normal measures satisfy that the restriction of their ultrapower embeddings to the ground model is an iterated ultrapower of the core model [30, 37, 39] — such iterated ultrapowers are particularly well understood, thanks to the nature of the core model. Second, the Friedman–Magidor forcing incorporates a coding mechanism that prevents unwanted normal measures from being created, and in turn this coding utilizes the fine structure of the ground model

This fine-structural approach, if elegant, imposes serious limitations on the extent of the applicability of Friedman–Magidor's methods. Namely, they are limited to those large cardinals within the reach of current inner model theory. For instance, the methodology cannot be leveraged to say anything about the number of normal measures on the first strongly compact cardinal, let alone on the first measurable limit of supercompact cardinals.

In response to this, work of Gitik and the second author [17] suggested an alternative route to bypass these limitations. Their methods make it possible to show, for instance, that the first strongly compact cardinal can carry exactly η normal measures $\langle U_\tau : \tau < \eta \rangle$, where each U_τ has Mitchell order τ, and there are no other measurable cardinals with Mitchell order greater than or equal to η [17, Theorem 4.16]. This is proved assuming the consistency of a supercompact cardinal with GCH and the linearity of the Mitchell order[3].

The proof technique involves performing a nonstationary support iteration of Prikry forcings (see [11]), adding a Prikry sequence to every measurable cardinal of Mitchell order $\geq \eta$ below the least supercompact cardinal κ. Under the additional assumption that there are no measurable cardinals above κ, it is shown that κ remains strongly compact in the generic extension, and that the normal measures on κ have the desired configuration.

This method was further refined in [17] to establish the consistency of the first strongly compact cardinal κ being the least measurable cardinal and carrying exactly η normal measures, for any given $\eta < \kappa$ (see [17, Theorem 4.17]). The argument begins with the model described above, in

[2]We concentrate here on the GCH context, under which the number of normal measures on a measurable cardinal κ is at most κ^{++}.

[3]In fact, weaker assumptions suffice; see [17, Theorem 4.16].

which the least strongly compact cardinal κ carries η normal measures, and proceeds by performing a Magidor (i.e., full support) iteration of Prikry forcings, adding a Prikry sequence to every measurable cardinal below κ. By a well-known theorem of Magidor [35], κ is strongly compact and the least measurable cardinal in the resulting model. As observed by Ben-Neria in [7], the Magidor iteration of Prikry forcings does not change the number of normal measures on κ, and hence κ carries exactly η normal measures in the generic extension. While Ben-Neria assumes that the Magidor iteration is performed over a canonical inner model, his result remains valid under the weaker assumption of GCH in the ground model (see [27]).

A natural question arising from the above is whether similar results can be obtained, for instance, at the second strongly compact cardinal. The methods of [17] encounter two main obstacles. First, anti–large cardinal assumptions are used to ensure that a supercompact cardinal κ remains strongly compact after a nonstationary support iteration below κ — it is assumed that there are no measurable cardinals above κ. Second, the least strongly compact cardinal κ ceases to be strongly compact after a Prikry sequence is added above it, preventing the use of similar methods to control the number of normal measures on the second strongly compact cardinal.

In [28], the second author introduced the splitting forcing, which circumvents the reliance on inner model theory present in the Friedman–Magidor forcing.[4] One of the principal advantages of the splitting forcing is its remarkably simple nature. This simplicity is largely due to suggestions of Omer Ben-Neria, whose insights significantly streamlined and improved the second author's original forcing construction. In the present paper, we employ variations of the splitting forcing to bypass the use of Prikry-type forcings in [17]. A key appeal of the splitting forcing is its flexibility: It can be modified, refined, and combined with other forcing techniques in a way that preserves many large cardinal configurations already present in the ground model.

Our theorems (see Theorems 3.1, 3.2, 3.3, 3.5, and 4.4 below) are proved relative to the assumption that the relevant large cardinals are consistent with GCH and Goldberg's Ultrapower Axiom (UA) (see [19] for details on UA). In fact, the proofs use only one consequence of UA: The existence of a unique normal measure of Mitchell order 0 on every measurable cardinal.[5] Thus, it suffices to assume that this property is consistent with the relevant large cardinal hypotheses. The plausibility of our initial assumptions is strongly supported by Goldberg's work, which includes a detailed analysis of the structure of the universe under UA in the presence of very large cardinals.

Building upon these ideas, we prove the following:

Theorem 3.1. *Assume the* GCH *holds, there are $n < \omega$ supercompact cardinals $\langle \kappa_i : i < n \rangle$, there are no measurable cardinals above κ_{n-1}, and each κ_i has a unique normal measure of Mitchell order* 0*. For every $i < n$, let $\tau_i \le \kappa_i^{++}$ be a cardinal. Then there is a generic extension where* GCH *holds, $\langle \kappa_i : i < n \rangle$ are the first n strongly compact cardinals, the first n measurable cardinals, and each κ_i carries exactly τ_i normal measures.*

Theorem 3.2. *Assume the* GCH *holds, κ is a supercompact cardinal, and λ is the first measurable cardinal above κ. In addition, suppose that λ has a unique normal measure. Then, for each cardinal $\tau \le \lambda^{++}$, there is a forcing extension where the following hold:*

(1) κ is supercompact;
(2) λ is measurable;
(3) λ carries exactly τ normal measures.

Theorem 3.3. *Assume the* GCH *holds, and suppose that κ is the first measurable limit of supercompact (strong) cardinals. Assume that κ carries a unique normal measure of Mitchell order 0. Let $\tau \le \kappa^{++}$ be a cardinal. Then, there is a generic extension where κ remains the first measurable limit of supercompact (strong) cardinals, and κ carries exactly τ normal measures.*

[4] The method has the additional virtue of bypassing the use of generalized Sacks forcing and self-coding forcings.

[5] This follows immediately from the linearity of the Mitchell order, which is a well-known consequence of UA (see [19, Theorem 2.3.11]).

Theorem 3.1 combines the classical Kimchi–Magidor identity crisis theorem with our new techniques for analyzing the number of normal measures. An important component in the original Kimchi–Magidor proof is its use of Laver's forcing which makes the supercompact cardinals indestructible under sufficiently directed-closed forcings. However, Laver's indestructibility preparation also introduces many unwanted normal measures to the generic extension; our analysis includes a new version of the indestructibility theorem, which grants control over the number of normal measures of trivial Mitchell rank in the generic extension (see Theorem 2.15).

As mentioned above, anti-large cardinal assumptions were used in the Gitik–Kaplan proof that the least measurable cardinal may be strongly compact and carry a unique normal measure. In an unpublished work, the same result was already established by Goldberg–Woodin relative to the consistency of UA with a measurable limit of supercompact cardinals, without appealing to any anti–large cardinal assumption. Building on that, and relying on Theorem 3.3, we are able to prove the following strengthening of the original Goldberg–Woodin theorem:

Theorem 3.5. *Assume the* GCH *holds, κ is a measurable limit of supercompact cardinals, $\tau \leq \kappa^{++}$, and κ has a unique normal measure of Mitchell order 0. Then there is a generic extension where κ is the least measurable cardinal, the least strongly compact cardinal, and κ carries exactly τ normal measures.*

We close the paper with an improvement of a theorem of Goldberg, Onsinski and Poveda [20] on the status of the first supercompact cardinal in HOD, assuming Woodin's HOD hypothesis. Recall that under the HOD hypothesis, the first extendible cardinal (in fact, HOD-supercompact suffices) is supercompact in HOD [43, 42].[6] A natural question is if this is optimal – namely, whether or not the first supercompact cardinal must be supercompact in HOD, under the HOD hypothesis. This was answered in the negative in [20] by producing a model of the HOD hypothesis where the first supercompact cardinal κ is strongly compact yet not 2^κ-supercompact in HOD. Taking the ideas of [20] as a stepping stone, we produce a model of the HOD hypothesis where the first supercompact cardinal is strongly compact in HOD, and it carries exactly one normal measure. This strengthens the configuration obtained in [20].

Theorem 4.4. *Assume* GCH *and $V = \mathrm{gHOD}$ both hold, κ is a supercompact cardinal, and there exists a unique normal measure on κ with trivial Mitchell rank. Then, the following configuration is consistent:*

1. *The* HOD *hypothesis holds.*
2. *κ is supercompact.*
3. *In* HOD*, κ is strongly compact and it carries exactly one normal measure.*

2. Preliminaries

2.1. **Generalities on nonstationary support iterations.** The main technique used in this paper is nonstationary support forcing iterations.

We leverage this section to garner a few standard facts about these for later use. Recall that a set of ordinals A is called *nonstationary in inaccessibles* if for every inaccessible cardinal λ, $A \cap \lambda$ is nonstationary in λ.

Let κ be a Mahlo cardinal. An iterated forcing $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \colon \alpha < \kappa \rangle$ is a *nonstationary support iteration* if for every inaccessible cardinal $\alpha \leq \kappa$, $\mathbb{P}_\alpha$ is the nonstationary support limit of $\langle \mathbb{P}_\beta \colon \beta < \alpha \rangle$, and for every other value of $\alpha < \kappa$, $\mathbb{P}_\alpha$ is the inverse limit of $\langle \mathbb{P}_\beta \colon \beta < \alpha \rangle$. More pedantically, for every $\alpha \leq \kappa$, conditions $p \in \mathbb{P}_\alpha$ are functions with domain α, such that:

(1) For all $\beta < \alpha$, $p \restriction \beta \in \mathbb{P}_\beta$ and $p \restriction \beta \Vdash p(\beta) \in \dot{\mathbb{Q}}_\beta$.

[6] This result is best possible, in the sense that the first extendible cardinal may consistently be the first strongly compact cardinal in HOD [20].

(2) The support of p,
$$\mathrm{supp}(p) = \alpha \setminus \{\beta < \alpha \colon p \restriction \beta \Vdash p(\beta) = \mathbb{1}_{\dot{\mathbb{Q}}_\beta}\},$$
is nonstationary in inaccessibles.

Assume that κ is a Mahlo cardinal and $I \subseteq \kappa$ is a stationary set of inaccessible cardinals. We say that $\mathbb{P}$ is an *I-spaced nonstationary support iteration* if, in addition to being a nonstationary support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \colon \alpha < \kappa \rangle$, $\mathbb{P}$ satisfies that for every $\alpha \in \kappa \setminus I$, $\dot{\mathbb{Q}}_\alpha$ is forced by $\mathbb{P}_\alpha$ to be trivial forcing.

In our context, $\mathbb{P}$ will be a nonstationary support iteration whose iterates $\dot{\mathbb{Q}}_\alpha$ possess certain closure properties, such as *α-closure* or *α-strategic-closure*. Recall that a forcing poset $\mathbb{Q}$ is *α-closed* if any decreasing sequence of conditions in $\mathbb{Q}$ with length less than α admits a lower bound in $\mathbb{Q}$.

A natural weakening of α-closure that is very much prevalent in the literature is the notion of α-strategic closure:

Definition 2.1 (α-strategic closure)**.**

(1) Let $\mathbb{Q}$ be a forcing notion and α be a regular cardinal. Let $G_\alpha(\mathbb{Q})$ be the two player game consisting of α stages, in which the players construct a descending sequence of conditions. Player I plays at odd stages, and Player II plays at even stages (including limit stages). Player I wins if at some limit stage below α, Player II fails to pick a condition extending all the conditions constructed so far. Otherwise, Player II wins.
(2) We say that a forcing notion $\mathbb{Q}$ is *α-strategically-closed* if Player II has a winning strategy in the game $G_\alpha(\mathbb{Q})$.[7] Similarly, $\mathbb{Q}$ is *$<\alpha$-strategically closed* if for all $\beta < \alpha$, Player II has a winning strategy in $G_\beta(\mathbb{Q})$.

An instrumental tool in the context of nonstationary support iterations is the so-called *fusion lemma*. Roughly speaking, this lemma ensures (modulo some extra closure-type assumptions) that if $\mathbb{P}$ is a nonstationary support iteration and $\vec{d} = \langle d(\alpha) : \alpha < \kappa \rangle$ is a sequence of dense open sets of $\mathbb{P}$ then one can 'diagonalize' $\vec{D}$ on a club set of indices α.

Lemma 2.2 (The fusion lemma)**.** *Assume that κ is Mahlo and $I \subseteq \kappa$ is a stationary set of inaccessible cardinals. Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \colon \alpha < \kappa \rangle$ be an I-spaced nonstationary support iterated forcing, such that:*

(1) For every $\alpha < \kappa$, $\mathrm{rank}(\dot{\mathbb{Q}}_\alpha) < \min(I \setminus (\alpha + 1))$.
(2) For every $\alpha < \kappa$, $\mathbb{1} \Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ is α-strategically closed".
(3) For every $\alpha \in \kappa \setminus I$, $\mathbb{1} \Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ is trivial forcing".

Let $\mathbb{P}$ be the nonstationary support limit of $\langle \mathbb{P}_\alpha \colon \alpha < \kappa \rangle$. Then for every $p \in \mathbb{P}$ and a sequence $\vec{d} = \langle d(\alpha) \colon \alpha < \kappa \rangle$ of dense open subsets of $\mathbb{P}$, there exists a condition $p^* \leq p$ and a club $C \subseteq \kappa$ such that for every $\alpha \in C$,
$$\{r \in \mathbb{P}_{\alpha+1} \colon r^\frown (p^* \setminus \alpha + 1) \in d(\alpha)\}$$
is dense in $\mathbb{P}_{\alpha+1}$ below $p^* \restriction (\alpha + 1)$.

Proof. Fix for every $\alpha \in I$ a $\mathbb{P}_\alpha$-name $\dot{\tau}_\alpha$ for a strategy of Player II in the game $G_\alpha(\dot{\mathbb{Q}}_\alpha)$. We construct sequences:

- $\langle p_i \colon i < \kappa \rangle$ a decreasing sequence of conditions in $\mathbb{P}$.
- $\langle \alpha_i \colon i < \kappa \rangle$ a continuous, increasing cofinal sequence in κ.
- $\langle C_i \colon i < \kappa \rangle$ a decreasing sequence of club subsets of κ (with respect to inclusion), where each C_i is disjoint from $\mathrm{supp}(p_i)$.

The construction is done in such a way that the following properties hold:

- For every $i < j < \kappa$, $p_i \restriction \alpha_i + 1 = p_j \restriction \alpha_i + 1$.

[7] This notion is also referred to as $\prec \alpha$-strategic-closure in work of the first author with Shelah [6].

- For every $i < \kappa$, $\{r \in \mathbb{P}_{\alpha_i+1} \colon r^\frown p_i \setminus (\alpha_i + 1) \in d(\alpha_i)\}$ is dense in $\mathbb{P}_{\alpha_i+1}$ below $p_i \restriction (\alpha_i + 1)$.
- For every $i < j$, $\alpha_j \in C_i$.
- For every $i < \kappa$ and $\alpha \in \operatorname{supp}(p_i) \setminus (\alpha_i + 1)$, $p_i \restriction \alpha$ forces that $\langle p_j(\alpha) \colon j < i, \alpha \in \operatorname{supp}(p_j)\rangle$ is the sequence of moves of Player II in a partial run of the game $G_\alpha(\dot{\mathbb{Q}}_\alpha)$ in which Player II plays according to their winning strategy $\dot{\tau}_\alpha$.

Start the construction by letting $p_0 = p$, $\alpha_0 = 0$, and C_0 be any club disjoint from $\operatorname{supp}(p)$. Assume that p_i, α_i, C_i have been constructed for some $i < \kappa$. Let $\alpha_{i+1} = \min(C_i \setminus \alpha_i + 1)$. We define $p_{i+1} \leq p_i$ such that:

(1) $p_{i+1} \restriction \alpha_{i+1} = p_i \restriction \alpha_{i+1}$.
(2) $\alpha_{i+1} \notin \operatorname{supp}(p_{i+1})$ (and since $\alpha_{i+1} \in C_i$ and C_i is disjoint from $\operatorname{supp}(p_i)$, $p_{i+1} \restriction (\alpha_{i+1} + 1) = p_i \restriction (\alpha_{i+1} + 1)$).
(3) $p_{i+1} \setminus (\alpha_{i+1} + 1)$ satisfies the following two properties:
  (a) $p_{i+1} \setminus (\alpha_{i+1} + 1)$ is forced by $p_{i+1} \restriction (\alpha_{i+1} + 1)$ to be an extension $s \leq p_i \setminus (\alpha_{i+1} + 1)$ for which $\{r \in \mathbb{P}_{\alpha_{i+1}+1} \colon r^\frown s \in d(\alpha_{i+1})\}$ is a dense subset of $\mathbb{P}_{\alpha_{i+1}+1}$ below $p \restriction (\alpha_{i+1} + 1)$.
  (b) For every $\alpha \in \operatorname{supp}(p_{i+1}) \setminus (\alpha_{i+1} + 1)$, $p_{i+1} \restriction \alpha$ forces that $\langle p_j(\alpha) \colon j \leq i, \alpha \in \operatorname{supp}(p_j)\rangle$ is the sequence of moves of Player II in a run of the game $G_\alpha(\dot{\mathbb{Q}}_\alpha)$ in which Player II plays according to $\dot{\tau}_\alpha$.

Claim 2.2.1. *Clause (3) can be arranged.*

Proof of claim. We need to argue that such a condition s exists. This essentially follows from the fact that $|\mathbb{P}_{\alpha_{i+1}+1}|$ is strictly below the amount of strategic closure of $\mathbb{P} \setminus (\alpha_{i+1} + 1)$.[8] More formally, let $\langle q_\alpha \colon \alpha < \chi\rangle$ be an injective enumeration of all conditions in $\mathbb{P}_{\alpha_{i+1}+1}$ below $p_i \restriction (\alpha_{i+1} + 1)$. We construct $\mathbb{P}_{\alpha_{i+1}+1}$-names $\langle \dot{p}^\beta_{i+1} \colon \beta \leq \chi\rangle$ as follows:

- $p_{i+1} \restriction (\alpha_{i+1} + 1)$ forces that $\langle \dot{p}^\beta_{i+1} \colon \beta \leq \chi\rangle$ is a decreasing sequence of conditions in $\mathbb{P} \setminus (\alpha_{i+1} + 1)$.
- For every $\beta < \chi$, $q_\beta{}^\frown \dot{p}^\beta_{i+1} \in d(\alpha_{i+1})$.
- For every $\beta < \chi$ and $\alpha \in \operatorname{supp}(\dot{p}^\beta_{i+1}) \setminus (\alpha_{i+1} + 1)$, $\dot{p}^\beta_{i+1} \restriction \alpha$ forces that $\langle \dot{p}^\gamma_{i+1}(\alpha) \colon \gamma < \beta, \alpha \in \operatorname{supp}(\dot{p}^\gamma_{i+1})\rangle$ is the sequence of moves of Player II in a run of the game $G_\alpha(\dot{\mathbb{Q}}_\alpha)$ in which player II plays according to the strategy $\dot{\tau}_\alpha$.

Once such a sequence is constructed, we pick the desired $\mathbb{P}_{\alpha_{i+1}+1}$-name $\dot{s}$ to be a condition on $\mathbb{P} \setminus \alpha_{i+1} + 1$ with $\operatorname{supp}(\dot{s}) = \bigcup_{\beta \leq \chi} \operatorname{supp}(\dot{p}^\beta_{i+1})$, as follows: Assume that $\alpha \in (\alpha_{i+1}, \kappa)$ and $\dot{s} \restriction \alpha$ has been constructed. Since $\dot{s} \restriction \alpha$ extends $p_i \restriction (\alpha_{i+1} + 1, \alpha)$, it forces that $\langle p_j(\alpha) \colon j \leq i, \alpha \in \operatorname{supp}(p_j)\rangle$ is the sequence of moves of Player II in the game $G_\alpha(\dot{\mathbb{Q}}_\alpha)$ according to $\dot{\tau}_\alpha$. Assume that, in the same game, Player I picks the condition $\dot{p}^\chi_{i+1}(\alpha)$ at stage $i + 1$ (and, if $\alpha \notin \operatorname{supp}(\dot{p}^\chi_{i+1})$, we assume that Player I picks $\mathbb{1}_{\dot{\mathbb{Q}}_\alpha}$). Let $\dot{s}(\alpha)$ be the condition forced by $\dot{s} \restriction \alpha$ to be the response of Player II when playing according to $\dot{\tau}_\alpha$. Since α is an inaccessible cardinal strictly above α_{i+1} (and in particular, $\alpha > \chi$), the strategy $\dot{\tau}_\alpha$ ensures that Player II has a valid response at each stage in the run of the game described above (recall that $\dot{\tau}_\alpha$ witnesses that $\dot{\mathbb{Q}}_\alpha$ is α-strategically closed). This concludes the definition of the condition $\dot{s}$. Since $\chi < \min(I \setminus (\alpha_{i+1} + 1))$, the set $\operatorname{supp}(\dot{s})$ is forced to be nonstationary in inaccessibles of $I \setminus (\alpha_{i+1} + 1)$, so $\dot{s} \in \mathbb{P} \setminus (\alpha_{i+1} + 1)$. It is not hard to verify that it satisfies the requirements in clause (3) above.

Thus, it remains to construct the sequence $\langle \dot{p}^\beta_{i+1} \colon \beta \leq \chi\rangle$.

We first describe how the condition $\dot{p}^0_{i+1}$ is chosen. For that, pick a condition $q' \leq q_0{}^\frown(p_i \setminus (\alpha_{i+1} + 1))$ such that $q' \in d(\alpha_{i+1})$. Define $\dot{p}^0_{i+1}$ such that $\operatorname{supp}(\dot{p}^0_{i+1}) = \operatorname{supp}(q')$, and, for every $\alpha \in \operatorname{supp}(\dot{p}^0_{i+1})$, $\dot{p}^0_{i+1} \restriction \alpha$ forces that $\dot{p}^0_{i+1}(\alpha)$ is a name for the response dictated to Player II by $\dot{\tau}_\alpha$ after Player I picked $q'(\alpha)$ on their first move in the game $G_\alpha(\dot{\mathbb{Q}}_\alpha)$.

[8]Indeed, the argument we provide below shows that for every $\alpha < \kappa$, the forcing $\mathbb{P} \setminus (\alpha + 1)$ is $\min(I \setminus (\alpha + 1))$-strategically closed.

Assume now that $\beta \leq \chi$ and $\langle \dot{p}^{\gamma}_{i+1} \colon \gamma < \beta \rangle$ have been defined. Let $\dot{t} \in \mathbb{P} \setminus (\alpha_{i+1}+1)$ be inductively constructed, as follows: For every $\alpha \in (\alpha_{i+1}, \kappa)$, if $\dot{t} \restriction \alpha$ was defined, it forces that $\dot{t}(\alpha)$ is the move dictated to Player II by the strategy $\dot{\tau}_\alpha$ in a run of the game $G_\alpha(\dot{\mathbb{Q}}_\alpha)$ in which the moves of Player II were $\langle \dot{p}^{\gamma}_{i+1}(\alpha) \colon \gamma < \beta, \alpha \in \operatorname{supp}(\dot{p}^{\gamma}_{i+1}) \rangle$. Note that the condition $\dot{t} \in \mathbb{P} \setminus (\alpha_{i+1}+1)$ constructed this way is forced by $p_{i+1} \restriction (\alpha_{i+1}+1)$ to be a lower bound of $\langle \dot{p}^{\gamma}_{i+1} \colon \gamma < \beta \rangle$. Pick an extension $q' \leq (q_\gamma)^\frown \dot{t}$ such that $q' \in d(\alpha_{i+1})$. Now let $\dot{p}^{\beta}_{i+1} \in \mathbb{P} \setminus (\alpha_{i+1}+1)$ be such that, for every $\alpha \in (\alpha_{i+1}+1, \kappa)$, $\dot{p}^{\beta}_{i+1} \restriction \alpha$ forces that $\dot{p}^{\beta}_{i+1}(\alpha)$ is the move dictated to Player II by $\dot{\tau}_\alpha$ in the following run of the game $G_\alpha(\dot{\mathbb{Q}}_\alpha)$: The game begins with the run in which the moves of Player II were $\langle \dot{p}^{\gamma}_{i+1}(\alpha) \colon \gamma < \beta, \alpha \in \operatorname{supp}(\dot{p}^{\gamma}_{i+1}) \rangle^\frown \dot{t}(\alpha)$. Then Player I replies with the condition $q'(\alpha)$, and the response of Player II in its next round according to $\dot{\tau}_\alpha$ is taken to be $\dot{p}^{\beta}_{i+1}(\alpha)$.

This concludes the inductive construction of the sequence $\langle \dot{p}^{\beta}_{i+1} \colon \beta \leq \chi \rangle$, and therefore concludes also the construction of the condition $\dot{s}$.

□

Let us argue that the condition p_{i+1} constructed this way indeed has a nonstationary support. Fix an inaccessible $\lambda > \alpha_{i+1}$. We claim that there exists a club in λ that belongs to V, and is forced by $p_{i+1} \restriction (\alpha_{i+1}+1)$ to be disjoint from $\operatorname{supp}(p_{i+1} \setminus (\alpha_{i+1}+1))$. Indeed, there exists a $\mathbb{P} \restriction (\alpha_{i+1}+1)$-name $\dot{D}$ for a club in λ which is disjoint from $\operatorname{supp}(p_{i+1} \setminus (\alpha_{i+1}+1)$. By the fact that $\mathbb{P} \restriction (\alpha_{i+1}+1)$ is small relative to λ (and, in particular, is λ-c.c.), there exists a club $D^* \in V$ such that $p_{i+1} \restriction (\alpha_{i+1}+1)$ forces that D^* is contained in $\dot{D}$; in particular, D^* is disjoint from $\operatorname{supp}(p_{i+1})$.

Finally, let $C_{i+1} \subseteq C_i$ be a club subset of κ in V which is disjoint from the support of p_{i+1}. This concludes the successor step in the construction.

For limit steps in the construction, assume that $\langle p_j, \alpha_j, C_j \colon j < i \rangle$ were constructed for some $i < \kappa$. Let $\alpha_i = \sup_{j<i} \alpha_j$. We define $p_i \in \mathbb{P}$ such that:

(1) $p_i \restriction \alpha_i = \bigcup_{j<i} p_j \restriction (\alpha_j+1)$. Note that the union is increasing, and in the case where α_i is inaccessible, the sequence $\langle \alpha_j \colon j < i \rangle$ is a club in α_i disjoint from the $\operatorname{supp}(p_i \restriction \alpha_i)$. Thus, $p_i \restriction \alpha_i \in \mathbb{P}_{\alpha_i}$.

(2) $\alpha_i \notin \operatorname{supp}(p_i)$. In particular, $p_i \restriction (\alpha_i+1) \leq p_j \restriction (\alpha_i+1)$ for every $j < i$, since α_i lies outside $\operatorname{supp}(p_j)$ in that of a limit point of C_j.

(3) $p_i \setminus (\alpha_i+1)$ satisfies the following two properties:
   (a) $p_i \setminus (\alpha_i+1)$ is forced by $p_i \restriction (\alpha_i+1)$ to be a common extension $\dot{s}$ of $\langle p_j \setminus (\alpha_i+1) \colon j < i \rangle$ for which $\{r \in \mathbb{P}_{\alpha_i+1} \colon r^\frown \dot{s} \in d(\alpha_i)\}$ is a dense subset of $\mathbb{P}_{\alpha_i+1}$ below $p \restriction (\alpha_i+1)$.
   (b) For every $\alpha \in \operatorname{supp}(p_i \setminus (\alpha_i+1))$, $p_i \restriction \alpha$ forces that

$$\langle p_j(\alpha) \colon j \leq i, \alpha \in \operatorname{supp}(p_j) \rangle$$

   is the sequence of moves of Player II in a run of the game $G_\alpha(\dot{\mathbb{Q}}_\alpha)$ in which Player II plays according to $\dot{\tau}_\alpha$.

   The proof that such $\dot{s}$ can be constructed is similar to the proof of Claim 2.2.1. Note that for each coordinate $\alpha > \alpha_i$, the forcing $\dot{\mathbb{Q}}_\alpha$ is sufficiently strategically closed in order to pick local lower bounds.

Finally, let $p^* = \bigcup_{i<\kappa} p_i \restriction (\alpha_i+1)$. Clearly, $\operatorname{supp}(p^*) \cap \lambda$ is nonstationary in λ, for every inaccessible cardinal $\lambda < \kappa$. Also, $\operatorname{supp}(p^*)$ is nonstationary in κ, as witnessed by the club $C = \langle \alpha_i \colon i < \kappa \rangle$. To complete the proof of Lemma 2.2, we have to verify the following:

Claim 2.2.2. *For every $\alpha \in C$,*

$$\{r \in \mathbb{P}_{\alpha+1} \colon r^\frown (p^* \setminus \alpha+1) \in d(\alpha)\}$$

is dense in $\mathbb{P}_{\alpha+1}$ below $p^* \restriction (\alpha+1)$.

Proof of claim. Let $\alpha \in C$. Our inductive construction ensures that

$$D = \{r \in \mathbb{P}_{\alpha+1} \colon r \leq p_\alpha \restriction (\alpha+1) \wedge r^\frown p_\alpha \setminus (\alpha+1) \in d(\alpha)\}$$

is dense below $p_\alpha \restriction (\alpha+1) = p^* \restriction (\alpha+1)$.

Let $r \in D$ be arbitrary. Since $p^* \le p_\alpha$, in particular,

$$p^* \restriction (\alpha+1) \Vdash p^* \setminus (\alpha+1) \le p_\alpha \setminus (\alpha+1).$$

But then $r^\frown p^* \setminus (\alpha+1) \le r^\frown p_\alpha \setminus (\alpha+1)$, which yields

$$D \subseteq \{r \in \mathbb{P}_{\alpha+1} \colon r \le p^* \restriction (\alpha+1) \,\wedge\, r^\frown p^* \setminus (\alpha+1) \in d(\alpha)\},$$

and as a result the latter is dense below $p^* \restriction (\alpha+1)$. □

This concludes the proof of Lemma 2.2. □

Lemma 2.3. *Let $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \colon \alpha < \kappa\rangle$ be an I-spaced nonstationary support iteration satisfying the hypotheses of Lemma 2.2. Assume* GCH *and suppose that for every $\alpha \in I$, $\dot{\mathbb{Q}}_\alpha$ is forced by $\mathbb{P}_\alpha$ to be a forcing notion that preserves cardinals. Then, the following hold:*

(1) $\mathbb{P}$ preserves cardinals.

(2) Suppose that, in addition, for each $\alpha \in I$, $\dot{\mathbb{Q}}_\alpha$ is forced by $\mathbb{P}_\alpha$ to be a forcing notion that preserves GCH. *Then $\mathbb{P}$ preserves* GCH.

Proof. We first prove clause (1). By GCH, $|\mathbb{P}| = \kappa^+$. In particular, for every $\lambda \ge \kappa^{++}$, λ remains a cardinal. The fact that κ^+ remains a cardinal follows from the following claim, which relies on the previous fusion lemma:

Claim 2.3.1. *Let $\dot{f}$ be a $\mathbb{P}$-name such that $\mathbb{1} \Vdash_{\mathbb{P}}$ "$\dot{f} \colon \kappa \to \kappa^+$ is an increasing function". Then, the set $\{p \in \mathbb{P} : \exists \alpha < \kappa^+\, (p \Vdash_{\mathbb{P}} \mathrm{Im}(\dot{f}) \subseteq \alpha)\}$ is dense.*

In particular, forcing with $\mathbb{P}$ preserves κ^+.

Proof of claim. Let $\langle d(\alpha) : \alpha < \kappa\rangle$ be the sequence of dense open subsets of $\mathbb{P}$, where each $d(\alpha)$ consists of all the conditions $p \in \mathbb{P}$ which decide the value of $\dot{f}(\alpha)$. By Lemma 2.2, every condition can be extended to a condition p for which there exists a club $C \subseteq \kappa$ such that, for every $\alpha \in C$,

$$\{r \in \mathbb{P}_{\alpha+1} \colon r^\frown(p \setminus \alpha+1) \in d(\alpha)\}$$

is dense below $p \restriction (\alpha+1)$. In particular, let

$$\gamma = \sup\{\beta < \kappa^+ \colon \exists \alpha \in C\ \exists r \in \mathbb{P}_{\alpha+1}\ \Big(r^\frown(p \setminus \alpha+1) \Vdash \dot{f}(\alpha) < \beta\Big)\}.$$

Then $\gamma < \kappa^+$ (because $|\mathbb{P}_{\alpha+1}| < \kappa$) and clearly $p \Vdash \mathrm{Im}(\dot{f}) \subseteq \gamma$. □

In order to show that $\mathbb{P}$ preserves κ, it suffices to show that $\mathbb{P}$ preserves all cardinals $\lambda < \kappa$. Assume otherwise, and let $\lambda < \kappa$ be the least cardinal being collapsed by $\mathbb{P}$. Factor $\mathbb{P} = \mathbb{P}_\lambda * \dot{\mathbb{Q}}_\lambda * \mathbb{P} \setminus (\lambda+1)$. Since $\dot{\mathbb{Q}}_\lambda$ preserves cardinals and $\mathbb{P} \setminus (\lambda+1)$ is λ^+-distributive (see the proof of Claim 2.15.1), the forcing $\mathbb{P}_\lambda$ collapses λ. By the argument from the previous paragraph, λ cannot be a successor cardinal; this implies that λ is a limit cardinal, contradicting the assumption that it is the least cardinal collapsed by $\mathbb{P}$.

Let us proceed to clause (2). Let λ be a cardinal[9] and factor the iteration as $\mathbb{P} = \mathbb{P}_\lambda * \dot{\mathbb{Q}}_\lambda * \mathbb{P} \setminus (\lambda+1)$. In the ground model, "$2^\lambda = \lambda^+$" holds, so if $\mathbb{P}_\lambda$ were to preserve this fact we will be done – indeed, $\dot{\mathbb{Q}}_\lambda$ is forced to preserve the GCH and $\mathbb{P} \setminus (\lambda+1)$ is λ^+-distributive, ergo none of them change the power-set-function pattern at λ from the ground model. We prove that $\mathbb{P}_\lambda$ preserves "$2^\lambda = \lambda^+$" by induction on λ. Suppose that for every V-cardinal $\beta < \lambda$ forcing with $\mathbb{P}_\beta$ preserves the GCH at β. We will show that given any generic $G_\lambda \subseteq \mathbb{P}_\lambda$ and a set $X \in \mathcal{P}^{V[G_\lambda]}(\lambda)$ there is a club $C \in \mathrm{Cub}^V_\lambda$ and a sequence $\langle X_\beta \mid \beta \in C\rangle \in \prod_{\beta \in C} \mathcal{P}^{V[G_\beta]}(\beta)$ such that $X = \bigcup_{\beta \in C} X_\beta$.[10] Note that this set of sequences has size $(2^\lambda)^V = \lambda^+$: To show this notice that each sequence $\langle X_\beta \mid \beta \in C\rangle$ comes from a (partial) λ-sequence of $\mathbb{P}_\beta$-nice names τ_β, each naming a subset of β. Since for each of those β,

[9]Note that by the previous observation there is no ambiguity here.

[10]In the case where λ is V-singular, C will instead be the ordinal $\mathrm{cf}^V(\lambda)$.

$\tau_\beta \in V_\lambda$, we conclude that there are at most $|{}^\lambda V_\lambda|^V = \lambda^+$-many such sequences of names, ergo at most λ^+-many sequences $\langle X_\beta \mid \beta \in C\rangle$, and therefore at most λ^+-many $X \in \mathcal{P}^{V[G]}(\lambda)$.[11]

Case λ is regular: Let $X \in \mathcal{P}^{V[G_\lambda]}(\lambda)$. Without loss of generality let us assume that this is forced by the trivial condition. For each $\beta < \lambda$, let

$$d(\beta) := \{p \in \mathbb{P}_\lambda : p \restriction \beta + 1 \Vdash_{\mathbb{P}_{\beta+1}} \exists \dot{X}_\beta \subseteq \beta\, (p \setminus (\beta+1) \Vdash_{\mathbb{P}_\lambda/\dot{G}_{\beta+1}} \dot{X} \cap \beta = \dot{X}_\beta)\}.$$

This set is clearly open, and it is dense in that $\mathbb{P}_\lambda/\dot{G}_{\beta+1}$ is forced to be β^+-strategically-closed, by our assumptions in the lemma.

Invoking Lemma 2.2, we find a condition p^* and a club $C \subseteq \alpha$ such that, for each $\beta \in C$, the set $e(\beta) := \{r \in \mathbb{P}_{\beta+1} \mid r^\frown p^* \setminus (\beta+1) \in d(\beta)\}$ is dense below $p^* \restriction (\beta+1)$. Since there are $\mathbb{P}_\lambda$-dense many such conditions we can assume that $p^* \in G_\lambda$. Now, for each $\beta \in C$ and $r \in e(\beta)$, we invoke the Forcing Maximality Principle to let $\dot{X}_{\beta,r}$ a $\mathbb{P}_{\beta+1}$-name witnessing that $r \in e(\beta)$. Now we amalgamate those names into a single $\mathbb{P}_{\beta+1}$-name, taking

$$\dot{X}_\beta := \{\langle \dot{X}_{\beta,r}, r\rangle : r \in e(\beta)\}.$$

Clearly, $p^* \Vdash_{\mathbb{P}_\lambda} \dot{X}_\beta \subseteq \dot{X} \cap \beta$.[12] To show the converse, let $\gamma < \beta$ and $s \leq p^*$ forcing "$\gamma \in \dot{X} \cap \beta$". By density of $e(\beta)$ we may assume that $s \restriction (\beta+1) \in e(\beta)$. Therefore, $(s \restriction \beta+1)^\frown p^* \setminus (\beta+1) \in d(\beta)$ and thus the latter necessarily forces "$\gamma \in \dot{X}_{\beta, s\restriction(\beta+1)} \subseteq \dot{X}_\beta$", hence so does the stronger condition s as well. By a density argument this shows that $p^* \Vdash_{\mathbb{P}_\lambda} \dot{X} \cap \beta \subseteq \dot{X}_\beta$, as it was needed.

Case λ is singular in V: We consider the case where λ is a limit of members of $I \cap \lambda$ – in particular, λ is strong limit in V. The case where λ is not a limit of members of $I \cap \lambda$ is easier since then $\mathbb{P}_\lambda$ is forcing equivalent to the poset $\mathbb{P}_\gamma$ where $\gamma = \sup(I \cap \lambda) < \lambda$, and by a usual nice names argument and GCH in V this latter adds at most λ^+-many subsets to λ.

Denote $\zeta = \mathrm{cf}(\lambda)$. Consider the forcing $\mathbb{P}_\lambda \setminus (\zeta+1)$. It is ζ^+-strategically closed. Fix $\langle \lambda_\alpha \colon \alpha < \zeta\rangle$ an increasing cofinal sequence in λ such that $\lambda_0 > \zeta$. For each $\alpha < \zeta$, working in $V[G_\lambda]$, let us consider

$$d(\alpha) = \{q \in \mathbb{P}_\lambda/G_{\lambda_\alpha+1} \colon \exists A \subseteq \lambda_\alpha (q \Vdash_{\mathbb{P}_\lambda/G_{\lambda_\alpha+1}} \dot{X} \cap \lambda_\alpha = \dot{A})\}.$$

Each $d(\alpha)$ is a dense open subset of $\mathbb{P}_\lambda/G_{\lambda_\alpha+1}$, ergo the set

$$D(\alpha) = \{r \in \mathbb{P}_\lambda/G_{\zeta+1} \colon r \restriction \lambda_\alpha + 1 \Vdash \exists A_\alpha \subseteq \lambda_\alpha (r \setminus \lambda_\alpha + 1 \Vdash \dot{X} \cap \lambda_\alpha = A_\alpha)\}$$

is dense open in $\mathbb{P}_\lambda/G_{\zeta+1}$. This is true for every $\alpha < \zeta$.

Since $\mathbb{P}_\lambda/G_{\zeta+1}$ is ζ^+-strategically closed, the intersection $D^* = \bigcap_{\alpha<\zeta} D_\alpha$ is dense in $\mathbb{P}_\lambda/G_{\zeta+1}$. Work in $V[G_{\zeta+1}]$. If $q \in \mathbb{P}_\lambda/G_{\zeta+1}$ is a condition in the intersection D^* then q carries a sequence $\langle \dot{A}_\alpha \colon \alpha < \zeta\rangle$ such that each $\dot{A}_\alpha$ is a $\mathbb{P}_{\lambda_\alpha+1}/G_{\zeta+1}$-name for a subset of λ_α, and q basically reduces the name $\dot{X}$ to the sequence $\langle \dot{A}_\alpha \colon \alpha < \zeta\rangle$; to wit, $q \Vdash_{\mathbb{P}_\lambda/G_{\zeta+1}} \dot{A}_\alpha = \dot{X} \cap \lambda_\alpha$.

The above shows that the number of subsets of λ in $V[G_\lambda]$ is bounded by the number of sequences $(V_\lambda)^\zeta$ in $V[G_{\zeta+1}]$. Standard $\mathbb{P}_{\zeta+1}$-nice names arguments shows that the cardinality of this set is bounded by

$$\left(|(V_\lambda)^\zeta|^{|\mathbb{P}_{\zeta+1}|}\right)^V \leq |{}^{<\lambda}V_\lambda|^V = |\lambda^{\mathrm{cf}(\lambda)}|^V = \lambda^+.$$

This completes the proof of the lemma.

□

For later analysis we also record the proof of the following technical fact:

[11] In the case where λ is V-singular we will have a $\mathrm{cf}^V(\lambda)$-sequence of members of V_λ, which either way has size λ^+, by the GCH in the ground model.

[12] Of course, here we are identifying $\dot{X}_\beta$ with its natural lifting to a $\mathbb{P}_\lambda$-name.

Lemma 2.4. *Let $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \colon \alpha < \kappa \rangle$ be an I-spaced nonstationary support iteration satisfying the hypotheses of Lemma 2.2. Let $\dot{f} \colon \kappa \to \mathrm{Ord}$ be a $\mathbb{P}$-name for a function as forced by some condition $p \in \mathbb{P}$. Then, there is $p^* \leq p$, a club $C \subseteq \kappa$ and a function $F \colon \kappa \to V$ with $|F(\alpha)| \leq |\mathbb{P}_{\alpha+1}|^V$ for every $\alpha \in C$, such that $p^* \Vdash \forall \alpha \in \check{C}\, (\dot{f}(\alpha) \in \check{F}(\alpha))$.*

Proof. For each $\alpha < \kappa$ let $d(\alpha) := \{q \in \mathbb{P} \mid q \perp p \ \vee \ (\exists \beta \ q \Vdash \dot{f}(\alpha) = \beta)\}$. This is a dense open subset of $\mathbb{P}$. By the Fusion Lemma 2.2 there is $p^* \leq^* p$ and a club $C \subseteq \kappa$ such that $e(\alpha) := \{r \in \mathbb{P}_{\alpha+1} \mid r^\frown p^* \setminus (\alpha+1) \in d(\alpha)\}$ is dense below $p^* \restriction (\alpha+1)$ for all $\alpha \in C$. Consider $F \colon C \to V$ defined as

$$F \colon \alpha \mapsto \{\beta_r \mid r \in e(\alpha),\, r \leq p^* \restriction (\alpha+1),\, r^\frown p^* \setminus (\alpha+1) \Vdash \dot{f}(\alpha) = \beta_r\}.$$

It is clear that F is as wanted. □

Recall that an elementary embedding $j \colon V \to M$ is called an *extender ultrapower embedding* if there is an extender E such that j is the ultrapower embedding induced by E, j_E. For an overview of the general theory of extenders we refer the reader to Kanamori's book [26]. An important consequence of Lemma 2.2 is the lifting of extender ultrapower embeddings:

Lemma 2.5. *Let $I \subseteq \kappa$ be a stationary set consisting of inaccessible cardinals and $j \colon V \to M$ be an extender ultrapower embedding with $\mathrm{crit}(j) = \kappa$ for which the following properties hold:*

(ℵ) $V \vDash {}^\kappa M \subseteq M$.

(ℶ) *The generators of j are bounded below $j(h)(\kappa)$ for some function $h \colon \kappa \to \kappa$, which satisfies that $j(h)(\kappa) < \min(j(I) \setminus \kappa+1)$.*

Let $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \colon \alpha < \kappa \rangle$ be an I-spaced nonstationary support iteration of length κ satisfying the assumptions of Lemma 2.2. Let $G \subseteq \mathbb{P}$ be generic over V. Denote $\dot{\mathbb{Q}}_\kappa = j\left(\langle \dot{\mathbb{Q}}_\alpha \colon \alpha < \kappa \rangle\right)(\kappa)$. Then:

(1) $j(\mathbb{P})_\kappa = \mathbb{P}$, and $G \subseteq \mathbb{P}$ is generic over M.

(2) In $V[G]$, let $\mathbb{Q}_\kappa = (\dot{\mathbb{Q}}_\kappa)_G$. If $g \subseteq \mathbb{Q}_\kappa$ is generic over $V[G]$, then g is $\mathbb{Q}_\kappa$-generic over $M[G]$.

(3) $j[G] \setminus \kappa+1$ generates a $j(\mathbb{P}) \setminus \kappa+1$-generic set over $M[G * g]$; to wit,

$$H = \{q \in j(\mathbb{P}) \setminus \kappa+1 \colon \exists p \in G\ (q \geq j(p) \setminus \kappa+1)\}.$$

In particular, $j \colon V \to M$ lifts to an embedding $j^* \colon V[G] \to M[G * g * H]$ which is definable in $V[G * g]$. Moreover, if for each $\alpha \in I$ the forcing $\dot{\mathbb{Q}}_\alpha$ is forced to be either α^+-c.c. or α^+-closed, $M[G * g * H]$ is closed under κ-sequences in $V[G * g]$.

Proof. For clause (1), note that, for every $\alpha < \kappa$, $\mathbb{P}_\alpha \in V_\kappa$. Therefore $j(\mathbb{P})_\kappa$ is, over M, the nonstationary support limit of $\langle \mathbb{P}_\alpha \colon \alpha < \kappa \rangle$. The nonstationary support limit up to κ is computed correctly in M since M is closed under κ-sequences of its elements living in V.

Clause (2) holds since $M[G] \subseteq V[G]$.

For clause (3), assume that $E \subseteq j(\mathbb{P}) \setminus \kappa+1$ is a $\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa$-name for a dense open subset of $j(\mathbb{P}) \setminus \kappa+1$. Write $E = j(f)(a)$ for some set of generators $a \in [j(h)(\kappa)]^{<\omega}$ and some $f \colon [\kappa]^{|a|} \to V$. We can assume that $\kappa = \min(a)$ and, for every $\vec{\nu} \in \mathrm{dom}(f)$, $f(\nu)$ is a $\mathbb{P}_{\min(\vec{\nu})+1}$-name for a dense open subset of $\mathbb{P} \setminus (\min(\vec{\nu})+1)$. Define, for every $\alpha < \kappa$,

$$d(\alpha) = \{q \in \mathbb{P} \colon q \restriction \alpha \Vdash \forall \vec{\nu} \in [h(\alpha)]^{|a|}\ (q \setminus \alpha+1 \in f(\vec{\nu}))\}.$$

We claim that $d(\alpha)$ is a dense open subset of $\mathbb{P}$. For that, it suffices to prove that $\mathbb{P} \setminus (\alpha+1)$ is forced to be $|h(\alpha)|^+$-distributive. Indeed, $\mathbb{P} \setminus (\alpha+1)$ is forced to be $\min(I) \setminus (\alpha+1)$-strategically closed; this follows from the combination of the facts that the iteration is taken with respect to an I-spaced nonstationary support, and for each active forcing stage $\beta \in I \setminus (\alpha+1)$, $\dot{\mathbb{Q}}_\beta$ is forced to be β-strategically closed (see the proof of Claim 2.2.1 for a similar argument).

By the Lemma 2.2, there exists $p \in G$ and a club $C \subseteq \kappa$ such that, for every $\alpha \in C$,

$$\{r \in \mathbb{P}_{\alpha+1} \colon r^\frown p \setminus (\alpha+1) \in d(\alpha)\}$$

is dense below $p \restriction (\alpha+1)$. Clearly this yields

$$p \restriction \alpha+1 \Vdash \forall \vec{\nu} \in [h(\alpha)]^{|a|} \ (p \setminus (\alpha+1) \in f(\vec{\nu})).$$

Since $\mathbb{P}$ is the nonstationary support limit of $\langle \mathbb{P}_\alpha \colon \alpha < \kappa \rangle$, we can assume that the club C is disjoint from $\mathrm{supp}(p)$, by shrinking it if necessary. Since $C \subseteq \kappa$ is a club, we have $\kappa \in j(C)$. Therefore, $j(p) \restriction \kappa+1 = p^\frown \mathbb{1}_{\mathbb{Q}_\kappa} \in G * g$, and thus, in $V[G*g]$,

$$j(p) \setminus \kappa+1 \in j(f)(a)_{G*g} = (E)_{G*g}$$

as desired.

Since every $p \in G$ has a nonstationary support and H is generated by $j[G] \setminus \kappa+1$, we have that $j[G] \subseteq G*g*H$. By Silver's lifting criterion, j lifts to an elementary embedding $j^* \colon V[G] \to M[G*g*H]$.

Finally, let us assume that for every $\alpha \in I$, $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ is $\alpha^+ - c.c.$". We argue that $M[G*g*H]$ is closed under κ-sequences in $V[G*g]$.

Claim 2.5.1. *$M[G]$ is closed under κ-seuqneces in $V[G]$.*

Proof. Assume that $f \colon \kappa \to \mathrm{Ord}$ is a sequence in $V[G]$. Let $\dot{f}$ be a $\mathbb{P}$-name for it. By Lemma 2.2, there exists $p \in G$ and a club $C \subseteq \kappa$ such that for every $\alpha \in C$,

$$p{\restriction}_{\alpha+1} \Vdash \exists g_\alpha \in (Ord)^\alpha \ \left(p \setminus \alpha+1 \Vdash f \restriction \alpha = g_\alpha\right).$$

This is due to the fact that the forcings $\mathbb{P} \setminus (\alpha+1)$ are α^+-strategically closed, so the sets

$$d(\alpha) = \{q \in \mathbb{P} \setminus \alpha+1 \colon q \text{ decides } \dot{f} \restriction \alpha\}$$

are dense and open.

Next, for every $\alpha \in C$, fix a $\mathbb{P}_{\alpha+1}$-name $\dot{g}_\alpha$, forced by $p{\restriction}_{\alpha+1}$ to be the set $g_\alpha \in V^{\mathbb{P}_\alpha}$ above. Since M is closed under κ-sequences in V, $\langle \dot{g}_\alpha \colon \alpha \in C \rangle \in M$. Thus, $(\dot{f})_G$ can be reconstructed inside $M[G]$ from G and $\langle \dot{g}_\alpha \colon \alpha \in C \rangle$ as the union

$$\bigcup_{\alpha \in C} (g_\alpha)_{G \restriction \alpha+1}.$$

This concludes the proof of the claim. ∎

Next, we argue that $M[G*g]$ is closed under κ-sequences in $V[G*g]$. This is clear if $\mathbb{Q}_\kappa$ is κ^+-c.c. or κ^+-closed in $V[G]$ (for the κ^+-c.c. case, see [13, Proposition 8.4]). In turn, either of these properties hold true in $V[G]$ because $M[G]$ is closed under κ-sequences in $V[G]$, and the fact that $\mathbb{Q}_\kappa$ is either κ^+-c.c. or κ^+-closed in $M[G]$ by our assumption.

Finally, the fact that $M[G*g*H]$ is closed under κ-sequences in $V[G*g]$ follows immediately from the fact that $M[G*g]$ is closed under κ-sequences in $V[G*g]$. ∎

To conclude this preliminary section, we mention two theorems that will be key to our forcing analysis. The first is Laver's *Ground Model Definability Theorem*, and the second is Hamkins' *Gap Forcing Theorem*.

Theorem 2.6 (Laver's Ground Model Definability Theorem [33])**.** *Suppose that $\mathbb{P} \in V$ is a forcing notion and $G \subseteq \mathbb{P}$ is generic over V. Then V is a definable class of $V[G]$ from a parameter in V.*

Theorem 2.7 (Hamkins' Gap Forcing Theorem [22])**.** *Suppose that $\mathbb{P}$ is a forcing notion which has a gap at some cardinal δ; namely, $\mathbb{P}$ can be factored in the form $\mathbb{P} = \mathbb{P}_0 * \dot{\mathbb{P}}_1$, where $\mathbb{P}_0$ is nontrivial, $|\mathbb{P}_0| < \delta$ and $\Vdash_{\mathbb{P}_0}$ "$\dot{\mathbb{P}}_1$ is $(\delta+1)$-strategically closed." Let $G \subseteq \mathbb{P}$ be generic over V, and assume that $j^* \colon V[G] \to M^*$ is an elementary embedding with critical point $\kappa > \delta$, such that $M^* \subseteq V[G]$ and M^* is closed under δ-sequences of its elements that belong to $V[G]$. Then:*

1. *M^* has the form $M[H]$, where $M \subseteq V$ and $H = j^*(G)$ is $j^*(\mathbb{P})$-generic over M. Furthermore, $M = V \cap M[H]$.*
2. *If j^* is definable in $V[G]$ from parameters, then $j = j^* \restriction V \colon V \to M$ is definable in V from parameters.*

2.2. **The splitting forcing.** In this section we outline, for later use, a variation of the *splitting forcing* introduced by the second author in [28].

For the rest of this section, we fix a measurable cardinal κ, and for each $\eta < \kappa^+$ let $f_\eta \colon \kappa \to \kappa$ be the η-th canonical function.[13] To streamline notation, we also denote by f_{κ^+} the function $f_{\kappa^+} \colon \alpha \mapsto |\alpha|^+$.

Fix an ordinal $\tau < \kappa^+$ and $I \subseteq \kappa$ a stationary set consisting of inaccessible cardinals. Define an I-spaced nonstationary support iteration

$$\mathbb{P}^{\tau,I} = \langle \mathbb{P}^\tau_\alpha, \dot{\mathbb{Q}}^\tau_\alpha \colon \alpha < \kappa \rangle,$$

where, for each $\alpha < \kappa$, we distinguish between the following cases:

- If $\alpha \in I$, $\mathbb{P}^\tau_\alpha$ forces that $\dot{\mathbb{Q}}^\tau_\alpha$ is an atomic forcing of the form
$$\{\mathbb{1}_{\mathbb{Q}_\alpha}\} \cup f_\tau(\alpha),$$
where the ordinals in $f_\tau(\alpha)$ are declared to be an antichain in $\dot{\mathbb{Q}}^\tau_\alpha$.
- Else, $\dot{\mathbb{Q}}^\tau_\alpha$ is forced by $\mathbb{P}^\tau_\alpha$ to be trivial.

Given $G \subseteq \mathbb{P}^{\tau.I}$ generic over V, $\bigcup G$ naturally defines a function in $\prod_{\alpha<\kappa} f_\tau(\alpha)$, which maps each $\alpha < \kappa$ to the unique common value $p(\alpha) \in f_\tau(\alpha)$ for conditions $p \in G$ such that $\alpha \in \operatorname{supp}(p)$. Note that $\mathbb{P}^{\tau,I}$ satisfies (1)–(3) of Lemma 2.2. In particular, Lemma 2.5 will be applicable to $\mathbb{P}^{\tau,I}$ whenever we are presented with an appropriate extender ultrapower $j \colon V \to M$. Also, $\mathbb{P}^{\tau,I}$ has a gap at any large enough cardinal that is a successor of an inaccessible caridnal below κ.

The following theorem is a variation on one of the main results of [28].

Theorem 2.8. *Assume that κ is a measurable cardinal and $I \subseteq \kappa$ is a stationary set of inaccessible cardinals. Let $\tau \leq \kappa^+$ and $\mathbb{P} = \mathbb{P}^{\tau,I}$ be the above forcing. Then the following hold:*

(1) $\mathbb{P}$ preserves both κ and κ^+ as cardinals and it also preserves the measurability of κ. Assuming GCH, *$\mathbb{P}$ preserves cardinals.*

(2) For every normal measure $U \in V$ on κ,

  (a) If $I \in U$, there are τ distinct lifts of U to normal measures on κ in $V[G]$, $\langle U^*_\eta \colon \eta < \tau \rangle$, where, for every $\eta < \kappa$, U^*_η is generated in $V[G]$ from $U \cup \{S_\eta\}$, where
  $$S_\eta = \{\alpha < \kappa \colon \left(\bigcup G\right)(\alpha) = f_\eta(\alpha)\}.$$

  (b) If $I \notin U$, U generates a normal measure $U^* \in V[G]$ on κ in $V[G]$.

(3) For every normal measure $W \in V[G]$ on κ,

  (a) If $I \in W$, there exists $U \in V$ with $I \in U$, and some $\eta < \tau$, such that $W = U^*_\eta$.

  (b) If $I \notin W$, there exists $U \in V$ such that $I \notin U$ and $W = U^*$.

Remark 2.9. A situation of particular interest is when I is the set of all inaccessible cardinals below κ. In this case, the splitting forcing produces a generic extension in which every normal measure on κ has the form U^*_η for some normal measure $U \in V$ on κ and $\eta < \tau$. In particular, if the ground model carries a unique normal measure on κ (for instance, the ground model is $L[U]$), the generic extension has exactly τ normal measures on κ. This provides an alternative proof for the Friedman-Magidor Theorem ([14]), showing that, consistently from a measurable cardinal, for every $\tau \leq \kappa^{++}$ there exists a measurable cardinal κ with τ normal measures.[14]

Lemma 2.10. *Let $j \colon V \to M$ be an elementary embedding satisfying the assumptions of Lemma 2.5. Suppose that $\tau \leq \kappa^+$, and let $\mathbb{P} = \mathbb{P}^{\tau,I}$. Suppose $G \subseteq \mathbb{P}$ is generic over V. Then:*

(1) If $\kappa \in j(I)$, j lifts to $V[G]$ in exactly τ ways. That is,

  (a) for every $\eta < \tau$, there exists $H_\eta \subseteq j(\mathbb{P})$ generic over M and an elementary embedding $j^*_\eta \colon V[G] \to M[H_\eta]$ extending j.

[13] For the explicit definition of $\langle f_\eta \colon \eta < \kappa^+ \rangle$ we refer the reader to [25, Lemma 24.5].

[14] The case $\tau = \kappa^{++}$ follows from the Kunen-Paris Theorem (see [31]).

(b) *every elementary embedding* $j'\colon V[G]\to M'$ *that extends* j *has the form* j^*_η *for some* $\eta<\tau$ *(in particular,* $M'=M[H_\eta]$*).*

(2) *If* $\kappa\notin j(I)$*,* j *uniquely lifts to* $V[G]$*. That is,*

(a) *there exist* $H\subseteq j(\mathbb{P})$ *generic over* M *and an elementary embedding* $j^*_\eta\colon V[G]\to M[H]$ *extending* j*.*

(b) *every elementary embedding* $j'\colon V[G]\to M'$ *that extends* j *is equal to* j^* *(in particular,* $M'=M[H]$*).*

Proof. We verify all the clauses in turn.

(1a): Every $\eta<\tau$ defines a $V[G]$-generic set g_η for the poset $\mathbb{Q}_\kappa=j(\langle\mathbb{Q}_\alpha\colon\alpha<\kappa\rangle)(\kappa)$. Thus by Lemma 2.5, j lifts to $j^*_\tau\colon V[G]\to M[H_\eta]$, where $H_\eta=G*g_\eta*(j[G]\setminus\kappa+1)$.

(1b): By Laver's Ground Model Definability Theorem (Theorem 2.6), $H=j'(G)$ is $j(\mathbb{P})$-generic over $j'(V)$ and $j[G]\subseteq H$.

Here, by $j'(V)$ we mean the class definable in M' using the same formula that defines V inside $V[G]$, when j' is applied to the parameters appearing in the formula. We argue that $j'(V)=M$. Given an ordinal $\alpha\in\mathrm{Im}(j)$, write $\alpha=j(\beta)=j'(\beta)$ for some ordinal β, and note that $(V_\alpha)^{j'(V)}=j'(V_\beta)$ since V_β is the β-th rank initial segment of the ground V of $V[G]$. Since $j'\supseteq j$, $(V_\alpha)^{j'(V)}=j(V_\beta)$. It follows that

$$j'(V)=\bigcup_{\alpha\in\mathrm{Ord}}(V_\alpha)^{j'(V)}=\bigcup_{\beta\in\mathrm{Ord}}j(V_\beta)=M.[15]$$

Finally, let $\eta=j'(G)(\kappa)$. Then $\eta<j(f_\tau)(\kappa)=\tau$, and $H_\eta\subseteq H$. Therefore, by maximality of generic filters, $H=H_\eta$. Ergo, $j'=j^*_\eta$.

(2a): Note that $\mathbb{Q}_\kappa=j(\langle\mathbb{Q}_\alpha\colon\alpha<\kappa\rangle)(\kappa)$ is the trivial forcing in the case when $\kappa\notin j(I)$. In particular, by Lemma 2.5, $j[G]$ generates a $j(\mathbb{P})$-generic $H\subseteq j(\mathbb{P})$ set over M, and j lifts to $j^*\colon V[G]\to M[H]$.

(2b): As in the proof of clause (1b), $j'(G)$ is $j(\mathbb{P})$-generic over $j'(V)=M$ and contains $j[G]$. Since $j[G]$ already generates a generic $H\subseteq j(\mathbb{P})$ over M, we deduce that $j'(G)=H$, $M'=M[H]$, and $j'=j^*$. □

We are now in a position to prove Theorem 2.8:

Proof of Theorem 2.8. Let us prove each clause in turn:

(1). This was proved in Lemma 2.3 – note that the proof that κ and κ^+ remain cardinals does not rely on any GCH assumptions whatsoever.

(2). Given a normal measure $U\in V$ on κ, let $\langle j^*_\eta\colon\eta<\tau\rangle$ be all the lifts of j_{U_0} from Lemma 2.10. For each $\eta<\tau$, let U^*_η be the normal measure on κ derived from j^*_η using κ as a seed. Then each U^*_η is a normal measure on κ extending $U\cup\{S_\eta\}$, where

$$S_\eta=\{\alpha<\kappa\colon G(\alpha)=f_\eta(\alpha)\}.$$

In fact, $U\cup\{S_\eta\}$ generates (in $V[G]$) U^*_η for every $\eta<\tau$. Indeed, given $X\in U^*_\eta$, let $\dot{X}$ be a $\mathbb{P}$-name for it. Since $X\in U^*_\eta$, there exists $q\in H_\eta$ (where H_η is as in the notation of Lemma 2.10) such that $q\Vdash\check{\kappa}\in j_{U_0}(\dot{X})$. However, since $H_\eta=G*g_\eta*(j_{U_0}[G]\setminus\kappa+1)$, there exists $p\in G$ such that

$$p^\frown\{\langle\kappa,\eta\rangle\}^\frown(j_{U_0}(p)\setminus\kappa+1)\Vdash\check{\kappa}\in j_{U_0}(\dot{X}).$$

In particular,

$$B=\{\alpha<\kappa\colon p\restriction\alpha^\frown\{\langle\alpha,f_\eta(\alpha)\rangle\}^\frown(p\setminus\alpha+1)\Vdash\check{\alpha}\in\dot{X}\}\in U.$$

Thus, in $V[G]$, $B\cap S_\eta\subseteq X$.

(3). Let us argue now that every normal measure $W\in V[G]$ has the form U^*_η for some normal $U\in V$ and $\eta<\tau$. Given such W, denote by $j_W\colon V[G]\to M^*$ its ultrapower embedding. By

[15]Here we use that $j''\,\mathrm{Ord}$ is an unbounded subclass of Ord.

Hamkins' Gap Forcing Theorem, $j_W \restriction V$ is definable in V, since $\mathbb{P}$ has a gap below κ. Denote $U = W \cap V$. Then $U \in V$ since, in V, $U = \{X \subseteq \kappa \colon \kappa \in (j_W \restriction V)(X)\}$. Next, denote $\eta = j_W(G)(\kappa)$. Then, by the normality of W, $\eta < \tau$ and $S_\eta \in W$. It follows that $U \cup \{S_\eta\} \subseteq W$ and, since U^*_η is generated from $U \cup \{S_\eta\}$, $W = U^*_\eta$. $\square$

Finally, we argue that the splitting forcing preserves the Mitchell order.

Lemma 2.11 (Preservation of the Mitchell order)**.** *Let $I \subseteq \kappa$, $\tau < \kappa^+$ and $\mathbb{P} = \mathbb{P}^{\tau,I}$ be as in Theorem 2.8. Let $G \subseteq \mathbb{P}$ be generic over V.*

(1) Assume that U_0, U_1 are normal measures on κ in V. Let $W_0, W_1 \in V[G]$ be normal measures on κ with $U_0 \subseteq W_0$ and $U_1 \subseteq W_1$. Then,

$$V \models U_0 \lhd U_1 \iff V[G] \models W_0 \lhd W_1.$$

(2) For normal measures on κ, $U \in V$ and $W \in V[G]$, such that $U \subseteq W$,

$$(o(W))^{V[G]} = (o(U))^V.$$ [16]

Proof. Clause (2) is an immediate corollary of clause (1) and the facts that every normal measure on κ in V lifts to a normal measure on κ in $V[G]$, and every normal measure on κ in $V[G]$ is a lift of a normal measure on κ in V. Thus, we concentrate in proving clause (1). Let U_0, U_1, W_0, W_1 be as above. We first argue that

$$V \models U_0 \lhd U_1 \Longrightarrow V[G] \models W_0 \lhd W_1.$$

Assume first that $I \in U_0$, namely $W_0 = (U_0)^*_\eta$ for some $\eta < \tau$. Since G is $\mathbb{P}$-generic over M_{U_1} and $U_0 \in M_{U_1}$, $U_0 \cup \{S_\eta\}$ generates a normal measure $\left((U_0)^*_\eta\right)^{M_{U_1}[G]}$ in $M_{U_1}[G]$. It suffices to argue that $(U_0)^*_\eta = \left((U_0)^*_\eta\right)^{M_{U_1}[G]}$. For that, it would be enough to prove that

$$(\mathcal{P}(\kappa))^{V[G]} = (\mathcal{P}(\kappa))^{M_{U_1}[G]}$$

because the base of both measures is the same.

Assume that $X \in V[G]$ is a subset of κ, and fix a $\mathbb{P}$-name $\dot{X} \in V$ for it. By the Fusion Lemma 2.2, we can find $p \in G$ and a club $C \subseteq \kappa$ such that, for every $\alpha \in C$,

$$p \restriction (\alpha+1) \Vdash \exists X_\alpha \subseteq \alpha \left(p \setminus (\alpha+1) \Vdash \dot{X} \cap \alpha = X_\alpha\right).$$

In particular, for every $\alpha \in C$, there exists a witnessing $\mathbb{P}_{\alpha+1}$-name $\dot{X}_\alpha \in V_\kappa$. Since $C, \langle \dot{X}_\alpha \colon \alpha \in C\rangle \in M_{U_1}$ and $p \in G$, we can reconstruct X in $M_{U_1}[G]$, as $X = \bigcup_{\alpha \in C} \left(\dot{X}_\alpha\right)_{G_\alpha} \in M_{U_1}[G]$.

Assume now that $I \notin U_0$. In this case, W_0 is generated by U_0 in $V[G]$. Since $U_0 \in M_{U_1}$, it suffices to prove that $(U_0)^* = ((U_0)^*)^{M_{U_1}[G]}$, which again follows from the fact that $(\mathcal{P}(\kappa))^{V[G]} = (\mathcal{P}(\kappa))^{M_{U_1}[G]}$.

We proceed now and prove that for every U_0, U_1, W_0, W_1 as in the formulation of the lemma,

$$V[G] \models W_0 \lhd W_1 \Longrightarrow V \models U_0 \lhd U_1.$$

By the analysis in the proof of Theorem 2.8, $\mathrm{Ult}(V[G], W_1)$ has the form $M_{U_1}[H]$ for $H \subseteq j_{U_1}(\mathbb{P})$-generic[17] over M_{U_1} with $H \cap \mathbb{P} = G$. Since $j_{U_1}(\mathbb{P}) \setminus \kappa$ is sufficiently closed, $W_0 \in M_{U_1}[G]$. Finally, $\mathbb{P}$ has a gap below κ, ergo $U_0 = W_0 \cap M_{U_1} \in M_{U_1}$, as desired. $\square$

[16] Note that this is an equation between ordinals, and it remains true regardless of cardinals being collapsed in $V[G]$.

[17] Indeed, we showed that either $W_1 = (U_1)^*_\eta$ (if $I \in U_1$, for some $\eta < \tau$), or $W_1 = (U_1)^*$ (if $I \notin U_1$). In each case, the ultrapower embedding associated with W is an embedding $j^* \colon V[G] \to M_{U_1}[H]$ for some $H \subseteq j_{U_1}(\mathbb{P})$, as proved in Theorem 2.8.

Corollary 2.12. *Let λ be a supercompact cardinal, and denote by $\kappa < \lambda$ the least measurable cardinal. Assume that κ has a unique normal measure.*[18] *Fix $\tau \leq \kappa^+$. Then, in a forcing extension, λ is supercompact, κ is a measurable cardinal and it carries exactly τ normal measures.*

Proof. Let I be the set of all inaccessible cardinals below κ, and let $\mathbb{P} = \mathbb{P}^{\tau,I}$ be the splitting forcing (defined as an iterated forcing of length κ). Since $\mathbb{P}$ is small relative to λ, λ remains supercompact in $V[G]$, for $G \subseteq \mathbb{P}$ generic over V. By Theorem 2.8, κ carries exactly τ normal measures in $V[G]$. □

For later use (see e.g., Theorem 3.1) we prove the following technical lemma showing that the splitting forcing $\mathbb{P}^{\tau,I}$ preserves supercompact cardinals modulo a suitable choice of I:

Lemma 2.13. *Assume* GCH. *Let κ be a supercompact cardinal and fix $\kappa_0 < \kappa$. Let I be the set of inaccessible cardinals in (κ_0, κ) that are limit of strong cardinals. Fix $\tau \leq \kappa^+$. Denote $\mathbb{P} = \mathbb{P}^{\tau,I}$. Then*

(1) $\mathbb{P}$ is $(\kappa_0)^+$-directed closed, and $\mathbb{P}$ preserves cardinals and the GCH.
(2) $\mathbb{P}$ preserves the supercompactness of κ.
(3) Let $G \subseteq \mathbb{P}$ be generic over V. Then every normal measure $U \in V$ on κ lifts in τ ways to measures $\langle U^*_\eta : \eta < \tau \rangle$ on κ in $V[G]$. Furthermore, every normal measure $W \in V[G]$ on κ is such a lift.

Proof. Clearly, $\mathbb{P}$ is $(\kappa_0)^+$-directed closed. The fact that $\mathbb{P}$ preserves cardinals and GCH follows from Lemma 2.3. Let us proceed to clause (2).

Let $G \subseteq \mathbb{P}$ a V-generic. For any singular strong limit cardinal $\lambda > \kappa$ with $\mathrm{cf}(\lambda) > \kappa$ we show that κ is λ-supercompact in $V[G]$. Indeed, let $j : V \to M$ be the ultrapower embedding derived from a fine, normal measure on $\mathcal{P}_\kappa(\lambda)$ of Mitchell order 0; in particular, κ is not λ-supercompact in M.

Because λ is a strong limit cardinal, it follows that κ is $< \lambda$-supercompact in M. In particular, there are no strong cardinals in the interval (κ, λ) in M, for if there were a strong cardinal μ with $\kappa < \mu < \lambda$ in M then κ would be μ-supercompact in M, hence fully supercompact therein, thus contradicting our choice of the measure U.

This observation ensures that the forcing $j(\mathbb{P}) \restriction (\kappa, \lambda]$ is trivial in M.

Let us lift j by constructing a generic for $j(\mathbb{P})$ over M. Note that

$$j(\mathbb{P}) \restriction (\kappa + 1) = \mathbb{P} * \dot{\mathbb{Q}}_\kappa,$$

where $\dot{\mathbb{Q}}_\kappa$ is forced to be an atomic forcing that picks an ordinal $\eta < \tau$. Also $j(\mathbb{P}) \setminus (\kappa + 1)$ is λ^+-directed closed since there are no inaccessible cardinals that are limit of strong cardinals in the interval $(\kappa, \lambda]$.

Fix some $\eta < \tau$ (say, $\eta = 0$) and let $G * g_\eta$ be the corresponding $j(\mathbb{P}) \restriction (\kappa + 1)$-generic set over M. Since M is closed under λ sequence in V and $|\mathbb{P}| < \lambda$, $M[G] = M[G * g_\eta]$ is closed under λ-sequences in $V[G]$. Note that $j[G] \in M[G]$ since $j \restriction \mathbb{P} \in M$. Let

$$s = \bigcup j[G] \setminus (\kappa + 1) = \bigcup \{ j(p) \setminus (\kappa + 1) : p \in G \}.$$

Then $s \in j(\mathbb{P}) \setminus (\kappa + 1)$ is a master condition. Construct, in $V[G]$, a generic $H \subseteq j(\mathbb{P}) \setminus (\kappa + 1)$ with $s \in H$. This construction is possible because $V[G]$ has a list of order type $|j(\kappa)^{++,M}| = \lambda^+$ of all dense subsets of $j(\mathbb{P}) \setminus (\kappa + 1)$,[19] and $j(\mathbb{P}) \setminus (\kappa + 1)$ is λ^+-closed in $V[G]$. It follows that $j : V \to M$ lifts to $j^* : V[G] \to M[G * g_\tau * H]$, and that $M[G * g_\tau * H]$ is closed under λ-sequences inside $V[G]$. Therefore j^* witnesses that κ is λ-supercompact in $V[G]$. Since λ can be chosen arbitrarily high, κ is fully supercompact in $V[G]$.

Clause (3) follows from clause (2) of Theorem 2.8, and the following facts:

[18] This follows, for example, from the linearity of the Mitchell order.
[19] here we used GCH and the fact that $\mathrm{cf}(\lambda) > \kappa$.

(1) Every normal measure $U \in V$ on κ concentrates on I (this follows from κ being supercompact in V, and in particular, an inaccessible limit of strong cardinals).
(2) Every normal measure $W \in V[G]$ concentrates on I as well (this follows from κ being supercompact in $V[G]$ and the fact that $\mathbb{P}$ has a gap below κ, so every large enough strong cardinal below κ in $V[G]$ was already a strong cardinal in V. See also ([22, Corollary 13]).

This completes the proof of the lemma. □

Finally, we would like to adjust Lemma 2.13 to the case where $\tau = \kappa^{++}$.

Lemma 2.14. *Assume GCH. Let κ be a supercompact and fix $\kappa_0 < \kappa$. Let I be the set of inaccessible cardinals in (κ_0, κ) that are limit of strong cardinals. Let $\tau = \kappa^{++}$. Let $\mathbb{P} = \mathbb{P}^{\tau,I}$ be the I-spaced nonstationary support iteration*[20] *$\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \colon \alpha \leq \kappa\rangle$, where for every $\alpha \in I \cup \{\kappa\}$, $\dot{\mathbb{Q}}_\alpha = (Add(\alpha, 1))^{V^{\mathbb{P}_\alpha}}$ (note that a forcing is being done on κ). Then:*

(1) $\mathbb{P}$ is $(\kappa_0)^+$-directed closed, and $\mathbb{P}$ preserves cardinals and the GCH.
(2) $\mathbb{P}$ preserves the supercompactness of κ.
(3) Let $G \subseteq \mathbb{P}$ be generic over V. Every $U \in V$ lifts in κ^{++} ways to a normal measure on κ in $V[G]$. Furthermore, every normal measure $W \in V[G]$ is such a lift.

Proof. (Sketch) The facts that $\mathbb{P}$ is $(\kappa_0)^+$-directed closed, preserves cardinals and preserves the GCH follow as in Lemma 2.13. The proof that $\mathbb{P}$ preserves the supercompactness of κ is also very similar to the proof from Lemma 2.13: For $\lambda > \kappa$ a strong limit cardinal, lift a λ-supercompactness embedding $j \colon V \to M$ first to $j^* \colon V[G_{<\kappa}] \to M[G * H]$, where $G = G_{<\kappa} * G(\kappa)$ is $\mathbb{P}$-generic over V, and $H \subseteq j(\mathbb{P}) \setminus (\kappa+1)$ is generic over $M[G]$ that contains a suitable master condition s as in Lemma 2.13. The main difference is that now we need to construct $h \in V[G]$ which is $j(\mathbb{P})(j(\kappa))$-generic over $M[G * H]$ with $j^*[G(\kappa)] \subseteq h$. For that, we regard $\bigcup g \in M[G * H]$ as a master condition, and pick any $h \in V[G]$ which is $(Add(j(\kappa), 1))^{M[G*H]}$-generic over $M[G * H]$ with $h \restriction \kappa = g$. This ensures that j^* lifts to $j^{**} \colon V[G] \to M[G * H * h]$, witnessing λ-supercompactness of κ in $V[G]$.

Finally, proceed to the classification of normal measures on κ in $V[G]$. Since $\mathbb{P}$ has a gap below κ, every normal measure $W \in V[G]$ on κ extends a normal measure $U \in V$ on κ. The fact that U lifts in κ^{++}-many ways follows by lifting $j_U \colon V \to M_U$ to $j^* \colon V[G] \to M[G * H * h]$, where:

- $H = j_U[G_{<\kappa}] \setminus (\kappa + 1)$.
- $h \subseteq Add(j(\kappa), 1)$ is generic over $M[G * H]$ with $h \restriction \kappa = g$.

Standard arguments (see [13, Proposition 8.1]) show that there are $2^{\kappa^+} = \kappa^{++}$ many such generics h over $M[G * H]$. Each h induces a lift $j^*_h \colon V[G] \to M_U[G * H * h]$ of M_U. The normal measures $\langle U^*_h \colon h \text{ as above}\rangle$ are pairwise distinct lifts of U, as each gives rise to a different ultrapower embedding j^*_h. Thus U lifts in κ^{++}-many ways.[21] □

2.3. Laver's indestructibility via nonstationary support iterations.

Next, we prove a version of Laver's theorem on making a supercompact cardinal indestructible under a certain class of forcings ([32]). The wrinkle here is that the preparatory forcing is an I-spaced nonstationary-support iteration instead of an Easton support iteration. This allows for a much finer control of certain normal measures on κ in the resulting generic extension.

Assume that κ is a supercompact cardinal. Let $\ell \colon \kappa \to V_\kappa$ be a Laver function for κ. Recall that this means that, for every $\lambda \geq \kappa$ and $x \in H_{\lambda^+}$, there exists a fine, normal ultrafilter $\mathcal{U}$ on $\mathcal{P}_\kappa(\lambda)$ such that $j_{\mathcal{U}}(\ell)(\kappa) = x$.

Theorem 2.15 (Laver indestructibility)**.** *Assume* GCH*, and let κ be a supercompact cardinal. There exists poset $\mathbb{P} \in V_{\kappa+2}$ such that whenever $G \subseteq \mathbb{P}$ is generic over V:*

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[20] An Easton support may be used here as well.

[21] We remark that there are other lifts, since $G(\kappa)$ doesn't have to be taken as the $Add(\kappa, 1)^{M[G]}$-generic over $M[G]$.

(1) κ is supercompact in $V[G]$, and its supercompactness is indestructible under κ^+-directed closed forcings.[22]

(2) Every normal measure $U \in V$ on κ of Mitchell order 0 generates a normal measure $U^* \in V[G]$ on κ of Mitchell order 0. Furthermore, every normal measure $W \in V[G]$ on κ of Mitchell order 0 has the form U^* for some normal measure $U \in V$ on κ of Mitchell order 0.

Proof. Our proof follows [13, Theorem 24.12].

Let $I \subseteq \kappa$ be the stationary set given by

$$\{\alpha < \kappa \colon \alpha \text{ is measurable and } \forall \beta < \alpha\, (\exists \gamma < \kappa\, \exists x \in V_\gamma\ \ (\ell(\beta) = (\gamma, x)) \to \gamma < \alpha)\}$$

We define an I-spaced nonstationary support iteration

$$\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \colon \alpha < \kappa \rangle$$

such that for every $\alpha < \kappa$:

- If α is inaccessible, $\mathbb{P}_\alpha$ is the nonstationary support limit of the iteration $\langle \mathbb{P}_\beta \colon \beta < \alpha \rangle$. Else, $\mathbb{P}_\beta$ is the inverse limit of $\langle \mathbb{P}_\beta \colon \beta < \alpha \rangle$.
- $\dot{\mathbb{Q}}_\alpha$ is forced to be trivial, unless $\alpha \in I$ and $\ell(\alpha)$ is a pair (γ, x) such that:
  - $\gamma < \min(I \setminus (\alpha + 1))$.
  - $x \in V_\gamma$ is a $\mathbb{P}_\alpha$-name for an α^+-directed closed poset.

  In that case, $\mathbb{P}_\alpha$ forces that $\dot{\mathbb{Q}}_\alpha$ is the poset x.

Let $\mathbb{P}$ be the nonstationary support limit of the forcings $\langle \mathbb{P}_\alpha \colon \alpha < \kappa \rangle$.

(1) Let $G \subseteq \mathbb{P}$ be generic over V. Let $\mathbb{Q} \in V[G]$ be a κ^+-directed closed poset, and $g \subseteq \mathbb{Q}$ generic over $V[G]$. We argue that κ is supercompact in $V[G * g]$ (in particular, κ is supercompact in $V[G]$ by taking $\mathbb{Q} = \{\mathbb{1}\}$).

Assume that $\lambda > \kappa$ bounds the rank of $\dot{\mathbb{Q}}$ in V. We argue that κ is λ-supercompact in $V[G*g]$. Let $\mu = 2^{2^\lambda}$, and let $\mathcal{U}$ be a supercompactness measure on $\mathcal{P}_\kappa(\mu)$ with $j_{\mathcal{U}}(\ell)(\kappa) = \left(\mu, \dot{\mathbb{Q}}\right)$. In particular, in $M_{\mathcal{U}}$, $\dot{\mathbb{Q}} = j_{U_0}(\langle \mathbb{Q}_\alpha \colon \alpha < \kappa \rangle)(\kappa)$ and $G * g$ is $j_{\mathcal{U}}(\mathbb{P}) \restriction (\kappa + 1)$-generic over $M_{\mathcal{U}}$. Furthermore, $\min(j_{\mathcal{U}}(I) \setminus (\kappa + 1)) > \mu$, so the forcing $j_{\mathcal{U}}(\mathbb{P}) \setminus (\kappa + 1)$ in $M_{\mathcal{U}}[G * g]$ is μ^+-directed-closed. Since $\mathbb{P} * \dot{\mathbb{Q}}$ is μ-c.c., $V[G*g] \vDash {}^{\mu}M_{\mathcal{U}}[G*g] \subseteq M_{\mathcal{U}}[G*g]$. Therefore, $j_{\mathcal{U}}(\mathbb{P}) \setminus (\kappa+1)$ is μ^+-directed-closed in $V[G * g]$ as well. Since the set $j_{\mathcal{U}}[G]$ belongs to $M_{\mathcal{U}}[G]$, we can find a master condition $q^* \in j_{\mathcal{U}}(\mathbb{P})$ such that for every $p \in G$, $q \leq j_{\mathcal{U}}(p)$ and $\kappa \notin \operatorname{supp}(q)$. Let $H \subseteq j_{\mathcal{U}}(\mathbb{P}) \setminus (\kappa + 1)$ be generic over $V[G * g]$ with $q^* \setminus (\kappa + 1) \in H$. Since $j_{\mathcal{U}}[G] \subseteq G * g * H$, we can lift the embedding $j_{\mathcal{U}} \colon V \to M_{\mathcal{U}}$ to $j' \colon V[G] \to M_{\mathcal{U}}[G * g * H]$, where j' is definable in $V[G * g]$. Finally, since $j'(\mathbb{Q})$ is $j_{\mathcal{U}}(\kappa^+)$-directed closed and $j'[g]$ has size at most $\mu \leq j_{\mathcal{U}}(\kappa^+)$ and belongs to $M_{\mathcal{U}}[G * g * H]$, we can find a master condition $r \in j'(\mathbb{Q})$, such that $r \leq j'(s)$ for every $s \in g$. Let $h \in V[G * g * H]$ be $j'(\mathbb{Q})$-generic over $M_{\mathcal{U}}[G * g * H]$. In $V[G * g * H * h]$, lift j' to $j^* \colon V[G] \to M_{\mathcal{U}}[G * g * H * h]$; this is possible since $j'[g] \subseteq h$.

Let $\mathcal{W} \in V[G * g * H * h]$ be the fine, normal measure on $(\mathcal{P}_\kappa(\lambda))^{V[G*g*H*h]}$ derived from j^* using $j^*[\lambda]$ as a seed. Since $j_{\mathcal{U}}(\kappa^+) > \mu$, the forcing $j'(\mathbb{Q})$ does not add new subsets to 2^{2^λ}. This, combined with the fact that $j_{\mathcal{U}}(\mathbb{P}) \setminus (\kappa+1)$ is μ^+-directed-closed in $V[G*g]$, implies that $\mathcal{W}$ already belongs to $V[G * g]$, so there exists a fine, normal measure on $(\mathcal{P}_\kappa(\lambda))^{V[G*g]}$ in $V[G * g]$. Since λ was arbitrary large, κ remains supercompact in $V[G * g]$.

(2) We prove this via two claims:

Claim 2.15.1. *Every normal measure $U \in V$ on κ of Mitchell order 0 generates a normal measure on κ in $V[G]$.*

[22] The standard version of the indestructibility theorem, in which the indestructibility is under κ-directed closed forcing, could also be proved, but it does not fulfill clause (2) above.

Proof of claim. Let $U \in V$ be a normal measure of order 0 on κ, and $j_{U_0} \colon V \to M_U$ be its ultrapower embedding. Note that $\kappa \notin j_{U_0}(I)$ since I consists of measurable cardinals and U has Mitchell order 0. Therefore $j_{U_0}(\langle \dot{\mathbb{Q}}_\alpha \colon \alpha < \kappa\rangle)(\kappa)$ is forced by $j_{U_0}(\mathbb{P})_\kappa = \mathbb{P}$ to be the trivial forcing. By Lemma 2.5, j_{U_0} lifts in $V[G]$ to an elementary embedding

$$j^* \colon V[G] \to M_U[j_{U_0}[G]]$$

witnessing the measurability of κ in $V[G]$.

Let U^* be the normal measure derived from j^* using κ as a seed.[23] Namely,

$$U^* = \left\{ X \in (\mathcal{P}(\kappa))^{V[G][H]} \colon \exists p \in G \left(j_{U_0}(p) \Vdash \check{\kappa} \in j_{U_0}(\dot{X}) \right) \right\}.$$

Given $X \in U^*$, there exists a $\mathbb{P}$-name $\dot{X}$ for X and a condition $p \in G$ such that $j_U(p) \Vdash \check{\kappa} \in j_U(\dot{X})$. In particular, the set $A = \{\alpha < \kappa \colon p \Vdash \check{\alpha} \in \dot{X}\}$ belongs to U, and since $p \in G$, $A \subseteq X$. Therefore, U^* is generated in $V[G]$ by U. □

Claim 2.15.2. *Every normal measure $W \in V[G]$ on κ of Mitchell order 0 is generated by a normal measure $U \in V$ on κ of Mitchell order 0.*

Proof of claim. Assume that $W \in V[G]$ is a normal measure on κ in $V[G]$ of Mitchell order 0. Denote the ultrapower embedding associated to W by

$$j_W^{V[G]} \colon V[G] \to M[H].$$

This is an elementary embedding, where $H = j_W(G)$ is $j_W(\mathbb{P})$-generic over some ground model M.

Since $\mathbb{P}$ has a gap below κ, Hamkins' Gap Forcing Theorem (Theorem 2.7) ensures that $U = W \cap V$ belongs to V and $M \subseteq V$. It suffices to argue that U has Mitchell order 0 in V; this will conclude the proof, since then W must be equal to the measure U^* generated by U in $V[G]$. Indeed, assume that U does not have Mitchell order 0 in V. Then U concentrates on I. In particular, $I \in W$. We will produce a contradiction by showing that every cardinal $\beta \in I$ remains measurable in $V[G]$, contradicting the fact that W has Mitchell order 0 in $V[G]$. Indeed, fix $\beta \in I$. By repeating the argument from Claim 2.15.1, every normal measure $U_\beta \in V$ on β of Mitchell order 0 generates a normal measure on β in $V[G_\beta]$. Since $\dot{\mathbb{Q}}_\beta$ is forced to be β^+-directed closed, the forcing $\mathbb{P} \setminus \beta$ does not add new subsets to β, so β remains measurable in $V[G]$, as desired. □

This concludes the proof of Theorem 2.15. □

A similar argument yields a version of the indestructibility theorem under forcings which preserve GCH.

Theorem 2.16 (Laver indestructibility with GCH and cardinal preservation)**.** *Assume* GCH*, and let κ be a supercompact cardinal. There exists poset $\mathbb{P} \in V_{\kappa+2}$ such that whenever $G \subseteq \mathbb{P}$ is generic over V:*

(1) κ is supercompact in $V[G]$, and its supercompactness is indestructible under κ^+-directed closed forcings which preserve cardinals and preserve GCH.
(2) Every normal measure $U \in V$ on κ of Mitchell order 0 generates a normal measure $U^* \in V[G]$ on κ of Mitchell order 0. Furthermore, every normal measure $W \in V[G]$ in κ of Mitchell order 0 has the form U^* for some normal measure $U \in V$ on κ of Mitchell order 0.
(3) GCH holds in $V[G]$, and every cardinal of V remains a cardinal in $V[G]$.

Proof. Let $\mathbb{P}$ be a poset similar to the one in the proof of Theorem 2.15, only requiring that whenever $\alpha \in I$ and $\ell(\alpha)$ is a pair (γ, x), $\dot{\mathbb{Q}}_\alpha = x$, provided that:

- $\gamma < \min(I \setminus (\alpha + 1))$.
- $x \in V_\gamma$ a $\mathbb{P}_\alpha$-name for an α^+-directed closed poset which preserves cardinals and GCH.

[23] It's not hard to prove that this extended embedding is the ultrapower embedding $j_{U^*}^{V[G]}$, where $U^* \in V[G]$ is the normal measure derived from it using κ as a seed.

Else, $\dot{\mathbb{Q}}_\alpha$ is trivial. Since $\mathbb{P}$ is now a nonstationary support iteration of cardinal preserving and GCH preserving posets, GCH holds in V in addition to the conclusions of Theorem 2.15. □

Notation 2.17. Fix the following notations for variations of the indestructibility preparations introduced in this section:

(1) Let κ be a supercompact cardinal, and let $\rho < \kappa$ be a regular cardinal. $\mathbb{L}(\kappa, \rho)$ is a ρ-directed closed version of the nonstationary support preparation that makes the supercompactness of κ indestructible under κ^+-directed-closed forcings preserving cardinals and the GCH.
(2) Let κ be a strong cardinal, and let $\rho < \kappa$ be a regular cardinal. $\mathbb{GS}(\kappa, \rho)$ is a ρ-directed closed version of the poset that makes the stronngness of κ indestructible under κ^+-directed-closed forcings. The existence of such a poset was proved by Gitik and Shelah (see [18]).[24]

Such forcings can easily be shown to exist by modifying the proofs of Theorem 2.16 and the Gitik-Shelah indestructibility preparation from [18] so that $\dot{\mathbb{Q}}_\alpha$ is forced to be trivial for every $\alpha < \rho$.

2.4. **Forcing nonreflecting stationary sets.** For every Mahlo cardinal α, we denote by $\mathbb{NR}(\alpha)$ the forcing notion that adds a nonreflecting stationary set to α. We use a modification of the standard forcing in which, for every regular cardinal $\gamma < \alpha$, there exists a dense subset $\mathbb{NR}_\gamma(\alpha) \subseteq \mathbb{NR}(\alpha)$ which is γ-directed closed. The forcings are defined as follows:

- $\mathbb{NR}(\alpha)$ consists of conditions which are functions $q \colon \beta \to 2$, where $0 < \beta < \alpha$, such that:
  - For every cardinal $\lambda \leq \beta$ with uncountable cofinality, there exists a club $C \subseteq \lambda$ such that $q \restriction C$ is identically 0.
  - For every inaccessible cardinal $\lambda < \beta$, the set $\{\alpha < \beta : \alpha \geq \lambda \wedge q(\alpha) = 1\}$ consists of ordinals of cofinality at least λ.

  Given $p, q \in \mathbb{NR}(\alpha)$, q extends p if q end-extends p.
- Given a regular cardinal $\gamma < \alpha$, $\mathbb{NR}_\gamma(\alpha)$ consists of conditions $q \in \mathbb{NR}(\alpha)$ such that $\gamma < \mathrm{dom}(q)$.

A generic $G \subseteq \mathbb{NR}(\alpha)$ over the ground model V induces a nonreflecting stationary subset of α (see Lemma 2.19 below).

Lemma 2.18. *Let α be a Mahlo cardinal, and let $\gamma < \alpha$ be a regular cardinal. Then $\mathbb{NR}(\alpha)$ and $\mathbb{NR}_\gamma(\alpha)$ are both α-strategically closed.*

Proof. We describe a winning strategy for Player II in the game $G_\alpha(\mathbb{NR}(\alpha))$ (a similar argument can be given for $\mathbb{NR}_\gamma(\alpha)$). Assume that $i < \alpha$ is an even stage and the players have constructed so far a descending sequence of conditions $\langle p_j : j < i \rangle \subseteq \mathbb{NR}(\alpha)$. Assume by induction the following properties:

- For every even $j < i$, $\mathrm{dom}(p_j)$ is an ordinal of the form $\beta_j + 1$, and $p_j(\beta_j) = 0$.
- $\langle \beta_j : j < i \text{ is even} \rangle$ is an increasing, continuous sequence.

If i is a limit ordinal, let $\beta_i = \sup_{j<i} \beta_j$. Let $p_i^* = \bigcup_{j<i} p_j$. The move of Player II in stage i will then be the condition $p_i = p_i^* \cup \{\langle \beta_i, 0 \rangle\}$.

If $i = i' + 1$ is a successor ordinal and $p_{i'}$ is the condition played by Player I in the most recent stage, Player II chooses at stage i an ordinal β_i strictly above $\mathrm{dom}(p_{i'})$, and plays with the condition $p_i \colon \beta_i + 1 \to 2$ such that $p_i \restriction \mathrm{dom}(p_{i'}) = p_{i'}$ and $p_i \restriction ((\beta_i + 1) \setminus \mathrm{dom}(p_{i'}))$ is identically 0. □

Lemma 2.19. *Assume that α is Mahlo. Let $G \subseteq \mathbb{NR}(\alpha)$ be generic over V. Let $S_G = \{\beta < \alpha \colon \exists p \in G\ (\beta \in \mathrm{dom}(p) \wedge p(\beta) = 1)\}$. Then S_G is a nonreflecting stationary subset of α.*

Proof. By the definition of $\mathbb{NR}(\alpha)$, S_G is nonreflecting; in fact, for every $\lambda < \alpha$ of uncountable cofinality, there exists a ground-model club in λ which is disjoint from S_G. Thus, let us concentrate on proving that S_G is stationary.

[24] In fact, the Gitik–Shelah indestructibility preparation applies to Prikry-type forcings with a sufficiently closed direct extension order; see [18] for more details. Let us also remark that a nonstationary support variation of the Gitik–Shelah indestructibility preparation exists, but will not be needed in the current paper.

Let $\dot{C}$ be an $\dot{\mathbb{NR}}(\alpha)$-name, forced by some condition $p \in \mathbb{NR}(\alpha)$ to be a club in α. Fix a winning strategy τ for Player II in the game $G_\alpha(\mathbb{NR}(\alpha))$ (such a strategy exists by Lemma 2.18). Let N be an elementary substructure of H_χ for some large enough χ, such that:

(1) $|N| < \alpha$.
(2) $\beta^* = \sup(N \cap \alpha)$ is an inaccessible cardinal below α.
(3) N is closed under $< \beta^*$-sequences of its elements.
(4) $\alpha, \mathbb{NR}(\alpha), p, \tau, \dot{C} \in N$.

Such N and β^* can be constructed by standard arguments, using the fact that α is Mahlo. Next, let us construct sequences:

- $\langle p_i : i < \beta^* \rangle \subseteq \mathbb{NR}(\alpha) \cap N$, a decreasing sequence of conditions.
- $\langle \beta_i : i < \beta^* \rangle \subseteq N \cap \alpha$ an increasing sequence of ordinals, which is unbounded in β^*, such that for every $i < \beta^*$,
$$p_{i+1} \Vdash \exists \delta \in (\beta_i, \beta_{i+1}) \cap \dot{C}.$$

We carry out the construction so that every strict initial segment of the above sequences belongs to N (although the sequences themselves lie in V). We also maintain the inductive assumption that for every $i < \beta^*$, the sequence $\langle p_j : j < i \rangle$ is the sequence of moves played by Player II in a partial run of the game $G_\alpha(\mathbb{NR}(\alpha))$, where Player II follows their winning strategy τ, and all conditions chosen so far by the players belong to N.

Let $p_0 = p$ and $\beta_0 = 0$. Suppose that $i < \beta^*$ and that $\langle p_j, \beta_j : j < i \rangle$ have been constructed. Since N is closed under $< \beta^*$-sequences of its elements, $\langle p_j : j < i \rangle \in N$, and it is the sequence of moves played by Player II in $G_\alpha(\mathbb{NR}(\alpha))$ according to τ.

If i is limit, Player II may follow τ to choose a lower bound $p_i \in N$ of $\langle p_j : j < i \rangle$. Let $\beta_i = \sup_{j<i} \beta_j$.

Assume $i = i' + 1$ is successor. Let $q \leq p_{i'}$ in N be such that q decides $\min\big(\dot{C} \setminus (\beta_{i'} + 1)\big)$. By elementarity, the decided value belongs to N. Let $\beta_i \in N \cap \alpha$ be a regular cardinal above both the decided value and $\max(\mathrm{dom}(q), \beta_{i'})$.[25] Extend q further to a condition q^* with domain $\beta_i + 1$ such that:

- $q^* \restriction \mathrm{dom}(q) = q$.
- $q^* \restriction [\mathrm{dom}(q), \beta_i) = 0$.
- $q^*(\beta_i) = 1$.

Note that $q^* \in N$ is a legitimate condition since β_i is a regular cardinal, so for every inaccessible cardinal $\lambda < \beta_i$, q^* assigns the value "1" only to ordinals of cofinality $\geq \lambda$. Finally, let Player I pick q^* as their next move in the game $G_\alpha(\mathbb{NR}(\alpha))$, and let $p_i \in N$ be the response of Player II when playing according to τ.

This concludes the inductive construction. Since $\langle p_i : i < \beta^* \rangle \in V$ consists of the β^* first rounds in a run of the game $G_\alpha(\mathbb{NR}(\alpha))$ according to τ, there exists a condition which extends each of the conditions $\langle p_i : i < \beta^* \rangle$. In particular, $p^* = \bigcup_{i<\beta^*} p_i$ is a condition in $\mathbb{NR}(\alpha)$. By our construction, p^* forces that β^* is a limit point of $\dot{C}$. Let $p^{**} = p^* \cup \{(\beta^*, 1)\}$. Then $p^{**} \in \mathbb{NR}(\alpha)$ since β^* is regular, p^{**} extends p, and $p^{**} \Vdash \beta^* \in \dot{C} \cap S_{\dot{G}}$. □

It will be useful that $\mathbb{NR}(\alpha)$ has arbitrarily closed dense subsets, a fact that it was also observed in [1].

Lemma 2.20. *Let $\gamma < \alpha$ be a regular cardinal. Then $\mathbb{NR}_\gamma(\alpha)$ is a γ-directed closed dense-open subset of $\mathbb{NR}(\alpha)$.*

Proof. The fact that $\mathbb{NR}_\gamma(\alpha)$ is a dense open subset of $\mathbb{NR}(\alpha)$ is clear. Let us argue that $\mathbb{NR}_\gamma(\alpha)$ is γ-directed closed. Assume $\delta < \gamma$ and $\langle q_i : i < \delta \rangle$ is a directed set of conditions in $\mathbb{NR}_\gamma(\alpha)$. Let $q^* = \bigcup_{i<\delta} q_i$. We argue that $q^* \in \mathbb{NR}_\gamma(\alpha)$ and from this it follows that q^* extends each of the

[25] By elementarity, N satisfies that every ordinal below α has a regular cardinal between it and α, so such a β_i exists.

conditions q_i for $i<\delta$. Let $\beta=\sup_{i<\delta}\operatorname{dom}(q_i)$. Clearly, $q^*\colon \beta\to 2$ is a well-defined function whose restriction to $S^{\beta}_{<\gamma}\setminus\gamma$ is identically 0. It remains to see that for every cardinal $\lambda\le\beta$ of uncountable cofinality, there exists a club $C\subseteq\lambda$ such that $q^*\restriction C$ is identically 0. This is clear if $\lambda\le\operatorname{dom}(q_i)$ for some $i<\delta$. Thus, it remains to check the case where $\lambda=\beta$ and $\operatorname{dom}(q_i)<\lambda$ for all $i<\delta$. In particular, $\lambda>\gamma$ and $\operatorname{cf}(\lambda)\le\delta$. Since $\delta<\gamma$, we can find a club $C\subseteq\lambda$ consisting of ordinals above γ, of cofinality strictly below γ. By the definition of the forcing, $q^*\restriction C$ is identically 0. Overall, this proves that $q^*\in\mathbb{NR}_\gamma(\alpha)$, as desired. □

Notation 2.21. Assume that $\kappa_0<\kappa$ and κ is Mahlo. Let $\mathbb{NR}(\kappa_0,\kappa)$ be the nonstationary support iterated forcing $\langle\mathbb{P}_\alpha,\dot{\mathbb{Q}}_\alpha\colon \alpha<\kappa\rangle$, where for $\alpha<\kappa$,

(1) If $\alpha\le\kappa_0$ or α is not measurable in V, $\dot{\mathbb{Q}}_\alpha$ is forced to be trivial.
(2) Else, $\dot{\mathbb{Q}}_\alpha$ is forced to be $(\mathbb{NR}(\alpha))^{V^{\mathbb{P}_\alpha}}$.

We summarize below the basic properties of the forcing $\mathbb{NR}(\kappa_0,\kappa)$:

Lemma 2.22. *Denote $\mathbb{P}=\mathbb{NR}(\kappa_0,\kappa)$. Then:*

(1) $\mathbb{P}$ is μ-strategically closed, being μ the first measurable above κ_0.
(2) $\mathbb{P}$ has a κ_0-directed closed dense subset.
(3) Assuming GCH*, $\mathbb{P}$ preserves cardinals and preserves* GCH*.*

Proof. Clause (1) follows, by an easy argument, from Lemma 2.18. Clause (2) follows from the fact that

$$D=\{p\in\mathbb{NR}(\kappa_0,\kappa)\colon \forall\alpha\in\operatorname{supp}(p)\ \Big(p\restriction\alpha\Vdash p(\alpha)\in\dot{\mathbb{NR}}_{\kappa_0}(\alpha)\Big)\}$$

is γ-directed closed by Lemma 2.20.[26] Clause (3) follows from Lemma 2.3. □

The following technical lemma will be useful in the proof of Theorem 2.24. It provides an example of a fusion argument, similar to that of Lemma 2.2, in which the dense sets $\langle d(\alpha)\colon \alpha<\kappa\rangle$ are not necessarily open.

Lemma 2.23. *Denote $\mathbb{P}=\mathbb{NR}(\kappa_0,\kappa)$. Let $D\subseteq\mathbb{P}$ be the set of conditions $p\in\mathbb{P}$ for which there exists a club $C\subseteq\kappa$ such that for every $\alpha\in C$ and for every $\beta\in\operatorname{supp}(p)\setminus(\alpha+1)$,*

$$p\restriction\beta\Vdash p(\beta)\in\dot{\mathbb{NR}}_{\alpha^{++}}(\beta).$$

Then D is a dense subset of $\mathbb{P}$.

Proof. The proof is a modified and simplified version of Lemma 2.2.[27] Fix a condition $p\in\mathbb{P}$. Let $C\subseteq\kappa$ be a club in κ disjoint from $\operatorname{supp}(p)$. Let us construct sequences:

- $\langle p_i\colon i<\kappa\rangle$ a decreasing sequence of conditions in $\mathbb{P}$.
- $\langle\alpha_i\colon i<\kappa\rangle$ a continuous, increasing cofinal sequence in κ, such that each α_i belongs to C.

The construction is done in such a way that the following properties hold:

- For every $i<\kappa$, $\operatorname{supp}(p_i)=\operatorname{supp}(p)$.
- For every $i<j<\kappa$, $p_i\restriction\alpha_i+1=p_j\restriction\alpha_i+1$.
- For every $i<\kappa$ and $\beta\in\operatorname{supp}(p)\setminus(\alpha_i+1)$, $p_i\restriction\beta$ forces that:
  - $p_i(\beta)\in\dot{\mathbb{NR}}_{(\alpha_i)^{++}}(\beta)$.
  - there exists $\gamma_i(\beta)\in(\alpha_i,\beta)$ such that $\operatorname{dom}(p_i(\beta))=\gamma_i(\beta)+1$ and $p_i(\beta)(\gamma_i(\beta))=0$.

Start the construction by letting $p_0=p$, $\alpha_0=\min(C)$. Assume that p_i,α_i have been constructed for some $i<\kappa$. Let $\alpha_{i+1}=\min(C\setminus\alpha_i+1)$. It's not hard to construct $p_{i+1}\le p_i$ such that:

- $\operatorname{supp}(p_{i+1})=\operatorname{supp}(p)$.

[26] Note that D is not open since coordinates can be added to the support when extending a condition.

[27] A naive attempt to deduce it directly from Lemma 2.2 would be to apply the lemma to the dense sets $d(\alpha)\subseteq\mathbb{P}$ consisting of conditions p such that for every $\beta\in\operatorname{supp}(p)\setminus(\alpha+1)$, $p\restriction\beta\Vdash p(\beta)\in\dot{\mathbb{NR}}_{\alpha^{++}}(\beta)$. However, these sets $d(\alpha)$ are not open.

- $p_{i+1} \restriction (\alpha_{i+1}+1) = p_i \restriction (\alpha_{i+1}+1)$.
- For every $\beta \in \mathrm{supp}(p) \setminus (\alpha_{i+1}+1)$, $p_{i+1} \restriction \beta$ forces that:
  - $p_{i+1}(\beta) \in \dot{\mathbb{NR}}_{(\alpha_{i+1})^{++}}(\beta)$.
  - There exists $\gamma_{i+1}(\beta) \in (\alpha_{i+1}, \beta)$ such that $\mathrm{dom}(p_{i+1}(\beta)) = \gamma_{i+1}(\beta)+1$, and $p_{i+1}(\beta)(\gamma_{i+1}(\beta)) = 0$.

Next, consider the limit case. Suppose that $i < \kappa$ and p_j, α_j were constructed for every $j < i$. Let $\alpha_i = \sup_{j<i} \alpha_j$. Define p_i such that:

- $\mathrm{supp}(p_i) = \mathrm{supp}(p)$.
- $p_i \restriction \alpha_i = \bigcup_{j<i} p_j \restriction \alpha_j$.
- $p_i \restriction (\alpha_i+1)$ forces that $p_i \setminus (\alpha_i+1)$ extends all the conditions $\langle p_j \setminus (\alpha_i+1) \colon j < i\rangle$.
- For every $\beta \in \mathrm{supp}(p) \setminus (\alpha_i+1)$, $p_i \restriction \beta$ forces that:
  - $p_i(\beta) \in \dot{\mathbb{NR}}_{(\alpha_i)^{++}}(\beta)$.
  - There exists an ordinal $\gamma_i(\beta) \in (\alpha_i, \beta)$ such that $\mathrm{dom}(p_i(\beta)) = \gamma_i(\beta)+1$ and $p_i(\beta)(\gamma_i(\beta)) = 0$.

Constructing p_i for i limit is not trivial as in the successor step, so let us justify why such a condition $p_i \in \mathbb{P}$ can be found. Note that $p_i \restriction \alpha_i$ is already fully determined, and α_i itself is outside the support of p_i. Thus, we concentrate on constructing $p_i \setminus (\alpha_i+1)$. Assume that $\beta \in \mathrm{supp}(p) \setminus (\alpha_i+1)$ and $p_i \restriction \beta$ has already been constructed. Work in a generic extension $V^{\mathbb{P}_\beta}$ that includes $p_i \restriction \beta$ in the generic, and let us construct $p_i(\beta)$ there. Note first that the sequence of conditions $\langle p_j(\beta) \colon j < i\rangle$ has a natural lower bound in $\mathbb{NR}(\beta)$,

$$x = \bigcup_{j<i} p_j(\beta).$$

We argue that x is a legitimate condition in $\mathbb{NR}(\beta)$. Denote $\gamma^* := \mathrm{dom}(x) = \sup_{j<i} \gamma_i(\beta)$. It suffices to prove that, for every regular cardinal $\lambda \le \gamma^*$ of uncountable cofinality, there exists a club $D \subseteq \lambda$ such that $x \restriction D$ is identically 0. Indeed, if $\lambda < \gamma^*$ this is clear, and if γ^* itself is regular and $\lambda = \gamma^*$, then $\langle \gamma_j(\beta) \colon j < i\rangle$ is a club in γ^* on which x vanishes.

Overall, $x \in \mathbb{NR}(\beta)$, and by further extending x inside $\mathbb{NR}(\beta)$, we can ensure that $x \in \mathbb{NR}_{(\alpha_i)^{++}}(\beta)$ and $\mathrm{dom}(x) = \gamma_i(\beta)+1$ for some $\gamma_i(\beta) \in (\alpha_i, \beta)$ with $p_i(\beta)(\gamma_i(\beta)) = 0$. Let $p(\beta)$ be forced by $p \restriction \beta$ to be the condition x constructed above.

This concludes the inductive construction. Finally, let

$$q = \bigcup_{i<\kappa} p_i \restriction \alpha_i \in \mathbb{P}.$$

Note that $\mathrm{supp}(q) = \mathrm{supp}(p)$ and q extends each of the conditions $\langle p_i \colon i < \kappa\rangle$. Furthermore, for every $i < \kappa$ and $\beta \in \mathrm{supp}(q) \setminus (\alpha_i+1)$,

$$q \restriction \beta \Vdash q(\beta) \le p_i(\beta) \in \mathbb{NR}_{(\alpha_i)^{++}}(\beta).$$

Thus, q is as desired, as witnessed by the club $\{\alpha_i \colon i < \kappa\}$. □

Theorem 2.24. *Assume* GCH. *Assume that κ is a supercompact cardinal, $\kappa_0 < \kappa$, and let $\mathbb{P} = \mathbb{NR}(\kappa_0, \kappa)$. Suppose that $\lambda > \kappa$ is a measurable cardinal, and there are no measurable cardinals in the interval (κ, λ). Then:*

(1) κ is λ-strongly compact after forcing with $\mathbb{NR}(\kappa_0, \kappa)$.
(2) There are no measurable cardianls in (κ_0, κ) after forcing with $\mathbb{NR}(\kappa_0, \kappa)$
(3) If λ is strongly compact, then κ remains so after forcing with $\mathbb{NR}(\kappa_0, \kappa)$.

For the proof of clause (3) in Theorem 2.24, we will use Ketonen's characterization of strongly compact cardinals, which reads as follows:

Theorem 2.25 (Ketonen [29]; Usuba [41, Lemmas 2.2 and 2.12])**.** *Let κ be a regular uncountable cardinal, and let $\lambda > \kappa$ be an inaccessible cardinal. Then the following are equivalent:*

(1) κ is λ-strongly compact.

(2) For every regular cardinal $\mu \in [\kappa, \lambda]$, there exists a κ-complete uniform ultrafilter[28] *on μ.*

Proof of Theorem 2.24. We begin by proving clause (1). Let W be a fine, normal measure on $\mathcal{P}_\kappa(\lambda)$. Fix normal measures, U_0 on κ and U_1 on λ, both of Mitchell order 0.

Perform the iterated ultrapower associated with the measure U_0, followed by the image of U_1, followed by the image of W. More formally, let:

- $j_{U_0}\colon V \to M_0 \simeq \mathrm{Ult}(V, U_0)$ be the ultrapower embedding associated with U_0. Denote $\kappa^* = j_{U_0}(\kappa)$.
- $j^{M_0}_{j_{U_0}(U_1)}\colon M_U \to M_1 \simeq \mathrm{Ult}(M_0, j_{U_0}(U_1))$ be the ultrapower embedding given by $j_{U_0}(U_1)$ over M_0.[29] Denote $\lambda^* = j^{M_0}_{j_{U_0}(U_1)}(\lambda)$.
- $j_1 = j^{M_0}_{j_{U_0}(U_1)} \circ j_{U_0}\colon V \to M_1$.
- $i = j^{M_1}_{j_1(W)}\colon M_1 \to M \simeq \mathrm{Ult}(M_1, j_1(W))$ the ultrapower embedding associated with $j_1(W)$ over M_1.
- $j = i \circ j_1\colon V \to M$.

Our goal is to show that j lifts, after forcing with $\mathbb{P} := \mathbb{NR}(\kappa_0, \kappa)$, to an elementary embedding that witnesses λ-strong compactness of κ in $V[G]$, where $G \subseteq \mathbb{P}$ is V-generic. Since there are no measurable cardinals in the interval (κ, λ) in V, the forcing $j(\mathbb{P})$ factors as follows in M:

$$j(\mathbb{P}) = j(\mathbb{P})_{\kappa^*} * \mathbb{NR}(\kappa^*) * \mathbb{NR}(\lambda^*) * \mathbb{NR}(\lambda^*, j(\kappa)).$$

We will construct generic objects for each of the factors.

Start with $j(\mathbb{P})_{\kappa^*}$. By Lemma 2.5, $j_{U_0}\colon V \to M_0$ lifts (inside $V[G]$) to

$$j_{U_0}\colon V[G] \to M_0[G * (j_{U_0}[G] \setminus \kappa + 1)].$$

(Note that the forcing used at stage κ of $j_{U_0}(\mathbb{P})$ is trivial, as U is assumed to have trivial Mitchell rank. This is why we obtain the above lifting).

Denote $H := G * (j_{U_0}[G] \setminus \kappa + 1)$. Clearly, H is $j(\mathbb{P})_{\kappa^*}$-generic over M_1 and M as well. Both $M_1[H]$ and $M[H]$ are closed under λ-sequence of their elements inside $M_0[H]$, and $M[H]$ is closed under λ^*-sequences in $M_1[H]$.

Proceed now with $\mathbb{NR}(\kappa^*)$. Let us construct $h_0 \in V[G]$ which is $\mathbb{NR}(\kappa^*)$-generic over the models $M_0[H], M_1[H]$ and $M[H]$. Note that

$$(\mathbb{NR}(\kappa^*))^{M_0[H]} = (\mathbb{NR}(\kappa^*))^{M_1[H]} = (\mathbb{NR}(\kappa^*))^{M[H]}$$

since $M_1[H]$ and $M[H]$ are closed under λ-sequences in $M_0[H]$.

For each $p \in G$, elementarity implies that $j(\mathrm{supp}(p))$ is nowhere stationary below $j(\kappa)$. In particular, $j(\mathrm{supp}(p)) \cap \kappa^*$ is nonstationary in κ^*, which entails $\kappa^* \notin j(\mathrm{supp}(p))$.[30] This observation indicates that there is no need to construct h_0 below a master condition; putted in different words, any $h_0 \in V[G]$ which is generic over $M[H]$ would work for lifting purposes. Additionally, let us observe that since both $M_1[H]$ and $M[H]$ are closed under λ-sequences in $M_0[H]$, it suffices to find such $h_0 \in V[G]$ which is generic over $M_0[H]$. We construct h_0 as follows: Recall that $M_0[H]$ is closed under κ-sequences inside $V[G]$, and thus, by Lemma 2.18, $(\mathbb{NR}(\kappa^*))^{M_0[H]}$ is κ^+-strategically closed in $V[G]$. Also, $V[G]$ lists in order-type κ^+ all dense subsets of $(\mathbb{NR}(\kappa^*))^{M_0[H]}$ inside the model $M_0[H]$. As a result, $h_0 \in V[G]$ can be constructed by meeting all those dense sets, one-by-one, in a run of the game $G_{\kappa^+}(\mathbb{NR}(\kappa^*))$ in which Player II plays according to their winning strategy. Note that both $M_1[H * h_0]$ and $M[H * h_0]$ remain closed under λ-sequences inside $M_0[H * h_0]$, and that $M[H * h_0]$ stays closed under λ^*-sequences in $M_1[H * h_0]$.

[28] An ultrafilter U on a regular cardinal μ is uniform if every set in U has size μ.

[29] $\lambda = j_{U_0}(\lambda)$ as $\lambda > \kappa$ is inaccessible. Thus $j_{U_0}(U_1)$ is a measure on λ in M_0.

[30] Indeed, let $C \subseteq \kappa^*$ be a club disjoint from $j(\mathrm{supp}(p)) \cap \kappa^* = \mathrm{supp}(p)$. Since $C \subseteq j(C)$ and $j(C)$ is a club on $j(\kappa^*)$, it follows that $\kappa^* \in j(C)$. Since $j(C)$ is disjoint from $j(\mathrm{supp}(p))$ we obtain the desired conclusion.

Next, proceed to $\mathbb{NR}(\lambda^*)$. Let us construct $h_1 \in M_0[H * h_0]$ which is $\mathbb{NR}(\lambda^*)$-generic over both $M_1[H * h_0]$ and $M[H * h_0]$. Once again, note that

$$(\mathbb{NR}(\lambda^*))^{M_1[H*h_0]} = (\mathbb{NR}(\lambda^*))^{M[H*h_0]}$$

as $M[H * h_0]$ is closed under λ^*-sequences in $M_1[H * h_0]$.

This part of the construction posses an addition caveat: It is not true anymore that $\lambda^* \notin \mathrm{supp}(j(p))$ for conditions $p \in G$. Therefore, in order to apply Silver's lifting criterion at the end of the construction, we will need to ensure that the generic h_1 is chosen so that $j[G] \restriction (\lambda^* + 1) \subseteq H * h_0 * h_1$. We do this by constructing a suitable master condition. The main obstacle is that the forcing $\mathbb{NR}(\lambda^*)$ is not sufficiently directed closed; by Lemma 2.20 it has a dense, ζ-directed closed subset for arbitrarily high $\zeta < \lambda^*$, but it's unclear that conditions of the form $i(q)(\lambda^*)$ for $q \in H$ make it into the relevant dense directed closed subset. We overcome this issue by isolating a dense subset $D \subseteq j_1(\mathbb{P})$, whose conditions are transferred through i to a sufficiently directed closed dense subset of $j(\mathbb{P})$.

By Lemma 2.23, there exists a dense subset $D \subseteq j_1(\mathbb{P})$ consisting of all conditions $q \in j_1(\mathbb{P})$ for which there exists a club $C \subseteq \kappa^*$, such that, for every $\alpha \in C$ and for every $\beta \in \mathrm{supp}(q) \setminus (\alpha + 1)$,

$$q \restriction \beta \Vdash q(\beta) \in \dot{\mathbb{NR}}_{\alpha^{++}, M_1}(\beta).$$

Given $q \in H \cap D$, let $C \subseteq \kappa^*$ be the club witnessing the fact that $q \in D$. Since $\mathrm{crit}(i) = \kappa^*$, $\kappa^* \in i(C)$. Thus, if $\lambda^* \in \mathrm{supp}(i(q))$, then

$$q \restriction \lambda^* \Vdash q(\lambda^*) \in \dot{\mathbb{NR}}_{(\kappa^*)^{++}, M}(\lambda^*).$$

Working in $M[H * h_0]$, consider the set

$$A = \{i(q)(\lambda^*) \colon q \in (H \cap D) \wedge \lambda^* \in \mathrm{supp}(i(q))\}.$$

Since $i \restriction j_1(\mathbb{P}) \in M$, $A \in M[H * h_0]$. Also $A \subseteq \mathbb{NR}_{(\kappa^*)^{++}}(\lambda^*)$ and $|A| = (\kappa^*)^+$. Since A is a directed set of conditions and $\mathbb{NR}_{(\kappa^*)^{++}}(\lambda^*)$ is a $(\kappa^*)^{++}$-directed closed poset, there exists in $M[H * h_0]$ a lower bound $t \in \mathbb{NR}(\lambda^*)$ of all the conditions in A. In particular, by the density of D, t is a lower bound of all the conditions in $\{i(q)(\lambda^*) \colon q \in H\}$. Since H is generated by conditions of the form $j_1(p)$[31] for $p \in G$, we deduce that t extends $j(p)(\lambda_*)$ for all $p \in G$. Finally, we construct $h_1 \in M_0[G * h_0]$ which is $\mathbb{NR}(\lambda^*)$-generic over both $M_1[H * h_0]$ and $M[H * h_0]$ with $t \in h_1$. To do this, note that $\mathbb{NR}(\lambda^*)$ is λ^{+,M_0}-strategically closed in the sense of $M_0[H * h_0]$, since $M_1[H * h_0]$ and $M[H * h_0]$ are closed under λ-sequences in $M_0[H * h_0]$. Furthermore, $M_0[H * h_0]$ has a list of order-type λ^{+,M_0} of all dense subsets of $\mathbb{NR}(\lambda^*)$. Thus, $h_1 \in M_0[H * h_0]$ can be constructed by meeting all those dense subsets, one-by-one, in a run of the game $G_{\lambda^{+,M_0}}(\mathbb{NR}(\lambda^*))$ in which Player II plays according to their winning strategy, and Player I opens the game by picking the condition t.

By standard arguments, $M[H * h_0 * h_1]$ remains closed under λ^*-sequences in $M_1[H * h_0 * h_1]$.

Finally, let us proceed with the last bit of the iteration, $\mathbb{NR}(\lambda^*, j(\kappa))$. Let us construct a suitable generic set $H^* \subseteq \mathbb{NR}(\lambda^*, j(\kappa))$ over $M[H * h_0 * h_1]$ such that $i[H] \setminus (\lambda^* + 1) \subseteq H^*$, where as usual

$$i[H] \setminus (\lambda^* + 1) := \{i(q) \setminus (\lambda^* + 1) \colon q \in H\}.$$

In order to construct H^*, we first need to construct a suitable master condition $s \in \mathbb{NR}(\lambda^*, j(\kappa))$, such that s extends all the conditions in $i[H] \setminus (\lambda^* + 1)$. We do this by the same technique that facilitated the construction of t above. Let $D \subseteq j_1(\mathbb{P})$ be the same dense set as above. Given $q \in H \cap D$, let $C \subseteq \kappa^*$ be the club witnessing the fact that $q \in D$. As above, $\kappa^* \in i(C)$, so for every $\beta \in \mathrm{supp}(i(q)) \setminus (\kappa^* + 1)$,

$$i(q) \restriction \beta \Vdash i(q)(\beta) \in \dot{\mathbb{NR}}_{(\kappa^*)^{++}}(\beta).$$

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[31] H was generated by conditions of the form $j_{U_0}(p)$. However, the critical point of $j_{U_0(U_1)}$ was way above the rank of $j_{U_0}(\mathbb{P})$ so $H = \{j_1(p) \mid p \in G\}$ is also generic.

In particular, $B = \{i(q) \setminus (\lambda^* + 1) \colon q \in H \cap D\}$ is contained in a $(\kappa^*)^{++,M}$-directed closed subset of $\mathbb{NR}(\lambda^*, j(\kappa))$. Since $|H| = (\kappa^*)^{+,M}$, there exists a lower bound $s \in \mathbb{NR}(\lambda^*, j(\kappa))$ of B. The same condition s is a lower bound of $i[H] \setminus (\lambda^* + 1)$ because D is dense in $j_1(\mathbb{P})$. Thus, it remains to construct $H^* \in M_1[H * h_0 * h_1]$ such that $H^* \subseteq \mathbb{NR}(\lambda^*, j(\kappa))$ is generic over $M[H * h_0 * h_1]$ and $s \in H^*$. Since $M[H * h_0 * h_1]$ is closed under λ^*-sequences inside $M_1[H * h_0 * h_1]$, $\mathbb{NR}(\lambda^*, j(\kappa))$ is $(\lambda^*)^{+,M_1}$-strategically closed in $M_1[H * h_0 * h_1]$. Also, $M_1[H * h_0 * h_1]$ has an enumeration of order type $(\lambda^*)^{+,M_1}$ consisting of all dense subsets of $\mathbb{NR}(\lambda^*, j(\kappa))$. Thus, the generic $H^* \in M_1[H * h_0 * h_1]$ can be constructed by meeting all those dense sets in a run of the game $G_{(\lambda^*)^{+,M_1}}(\mathbb{NR}(\lambda^*, j(\kappa)))$ in which Player II plays according to their winning strategy and Player I opens the game by picking the condition s. Finally, note that $M[H * h_0 * h_1 * H^*]$ remains closed under λ^*-sequences in $M_1[H * h_0 * h_1]$.

This concludes the construction of the generic extension $M[H * h_0 * h_1 * H^*]$ inside $V[G]$. Notice that we made sure through the construction that

$$j[G] \subseteq H * h_0 * h_1 * H^*.$$

Therefore, by Silver's lifting criterion, j lifts to an embedding

$$j^* \colon V[G] \to M[H * h_0 * h_1 * H^*].$$

It remains to prove that j^* witnesses λ-strong compactness of κ in $V[G]$. Indeed, this follows by combining the following facts:

- $M[H * h_0 * h_1 * H^*]$ is closed under κ-sequences in $V[G]$.
- $j(\kappa) > \lambda$.
- $j[\lambda]$ is covered by the set $S := i[j_1(\lambda)] = i[\lambda^*] \in M[H * h_0 * h_1 * H^*]$, and $|S|^{M[H*h_0*h_1*H^*]} = \lambda^* < j(\kappa)$.

For clause (2) of the theorem, observe that no cardinal is measurable if it carries a nonreflecting stationary subset. Hence every V-measurable cardinal in the interval (κ_0, κ) ceases to be measurable after forcing with $\mathbb{P}$. Moreover, any measurable cardinal in $V[G]$ must already have been measurable in V (Indeed, $\mathbb{P}$ has a gap at the least V-measurable cardinal and is trivial below it, so it cannot create new measurable cardinals). Thus, in $V[G]$, the interval (κ_0, κ) does not contain measurable cardinals.

Finally, let us prove clause (3) of the theorem. By Theorem 2.25, it suffices to prove that, in $V[G]$, every regular cardinal $\mu \geq \kappa$ has a κ-complete uniform ultrafilter. By clause (1), κ is λ-strongly compact in $V[G]$, so by Theorem 2.25, every regular cardinal $\mu \in [\kappa, \lambda]$ carries a κ-complete uniform ultrafilter in $V[G]$. Thus it remains to take care of regular cardinals above λ. Since λ is strongly compact in V, every regular cardinal $\mu > \lambda$ carries a uniform, λ-complete ultrafilter $U_\mu \in V$. Since $|\mathbb{P}| = \kappa^+$, the same ultrafilter generates a uniform, λ-complete ultrafilter U^*_μ on μ in $V[G]$,[32] as desired. □

The case where there are no measurable cardinals above κ will also matter to us, and it can be proved using the same techniques as above. In fact, the proof is simpler since there is no need to take an additional ultrapower with a normal measure of order 0 on λ.

Theorem 2.26. *Assume* GCH. *Assume that κ is a supercompact cardinal, $\kappa_0 < \kappa$, and let $\mathbb{P} = \mathbb{NR}(\kappa_0, \kappa)$. Suppose that there are no measurable cardinals above κ. Then κ is strongly compact after forcing with $\mathbb{NR}(\kappa_0, \kappa)$. Moreover, there are no measurable cardinals in the interval (κ_0, κ) after forcing with $\mathbb{NR}(\kappa_0, \kappa)$.*

[32]Indeed, for every $\eta < \lambda$ and a sequence $\langle A_i \colon i < \eta \rangle \subseteq U^*_\mu$ in $V[G]$, find a sequence $\langle B_i \colon i < \eta \rangle \subseteq U_\mu$ in $V[G]$ such that $B_i \subseteq A_i$ for all i. Let $S \in V$, $S \subseteq U_\mu$, $|S| < \lambda$ be a set that covers $\{B_i \colon i < \eta\}$. Then $\bigcap S \subseteq \bigcap_{i<\eta} A_i$ and $\bigcap S \in U_\mu$, so $\bigcap_{i<\eta} A_i \in U^*_\mu$.

Proof. Fix a strong limit cardinal $\lambda > \kappa$ with $\mathrm{cf}(\lambda) > \kappa$. Fix a fine, normal measure W on $\mathcal{P}_\kappa(\lambda)$. Let U be a normal measure on κ of Mitchell order 0. Consider the iterated ultrapower, first with U and then with the image of W, namely

$$j = j^{M_U}_{j_U(W)} \circ j_U \colon V \to M.$$

Note that $\lambda = j_U(\lambda)$ is not measurable under the hypotheses of the theorem, so there is no need to take an ultrapower with a normal measure of order 0 on λ. Denote $\kappa^* = j_U(\kappa)$, and factor

$$j(\mathbb{P}) = j(\mathbb{P})_{\kappa^*} * \dot{\mathbb{NR}}(\kappa^*) * \dot{\mathbb{NR}}(\lambda, j(\kappa)).$$

Now mimic the proof of Theorem 2.24 to find suitable generics $H * h_0 * H^*$ for $j(\mathbb{P})$ over M, and lift j to $j^*\colon V[G] \to M[H * h_0 * H^*]$, witnessing λ-strong compactness of κ in $V[G]$. We omit the details here since they are exactly the same as in the proof of Theorem 2.24.

Since λ was arbitrarily large, κ is fully strongly compact in $V[G]$. □

We conclude this subsection by characterizing normal measure on κ after forcing with $\mathbb{NR}(\kappa_0, \kappa)$.

Lemma 2.27. *Let κ be a measurable cardinal and $\kappa_0 < \kappa$. Denote $\mathbb{P} = \mathbb{NR}(\kappa_0, \kappa)$. Let $G \subseteq \mathbb{P}$ be generic over V. Then κ remains measurable in $V[G]$. Moreover:*

1. *For every normal measure $U \in V$ on κ of Mitchell order 0, U generates a normal measure $U^* \in V[G]$ on κ.*
2. *Every normal measure $W \in V[G]$ on κ has the form U^* for some normal measure $U \in V$ of Mitchell order 0 on κ.*

Proof. Let $U \in V$ a normal measure on order 0 on κ. Note that κ is not measurable in $M_U \simeq \mathrm{Ult}(V, U)$. Thus, by Lemma 2.5, j_{U_0} lifts to an embedding $j^*\colon V[G] \to M[G * (j_{U_0}[G] \setminus (\kappa + 1))]$. Define $U^* = \{X \subseteq \kappa \colon \kappa \in j^*(X)\}$. Then U^* is a normal measure on κ in $V[G]$. It's not hard to verify that j^* is the ultrapower embedding associated with U^*. Let us argue that U generates U^* in $V[G]$. Assume that $X \in V[G]$ is a subset of κ in U^*. Let $\dot{X}$ be a $\mathbb{P}$-name for X. Then, for some $p \in G$, $j_{U_0}(p) \Vdash \kappa \in j_{U_0}(\dot{X})$. It follows that $A = \{\alpha < \kappa \colon p \Vdash \alpha \in X\} \in U$, and $A \subseteq X$.

Conversely, assume that $W \in V[G]$ is a normal measure on κ. By Hamkins' Gap Forcing Theorem (Theorem 2.7), we have $U := W \cap V \in V$. It is straightforward to verify that $U \in V$ is a normal measure on κ. We claim that U must have Mitchell order 0. Let $\Delta \subseteq \kappa$ denote the set of measurable cardinals in V, and suppose toward a contradiction that $\Delta \in U$. Then in particular $\Delta \in W$. Recall that no cardinal in Δ remains measurable in $V[G]$, since $\mathbb{P}$ adds a nonreflecting stationary subset to each such cardinal. Since W is normal and $\Delta \in W$, it follows that $\kappa \in j_W(\Delta)$. Thus, in $\mathrm{Ult}(V[G], W)$, there exists a nonreflecting stationary subset $S \subseteq \kappa$. However, $V[G]$ and $\mathrm{Ult}(V[G], W)$ agree on subsets of κ and on their stationarity. Therefore, S is a nonreflecting stationary subset of κ in $V[G]$, contradicting the measurability of κ in $V[G]$.

It follows that U has Mitchell order 0. Finally, since $U \subseteq W$ and U generates an ultrafilter $U^* \in V[G]$, we deduce that $W = U^*$, as desired. □

3. The main theorems

In this section we will apply the theory developed so far to prove the main theorems of the paper. The general theme (as indicated in the introduction) is to produce models where measurable cardinals beyond the current reach of the Inner Model Program (e.g., measurable limits of supercompacts, etc.) carry any prescribed amount of normal measures. All of our results are proved under the assumption that in the ground model, each relevant measurable cardinal carries exactly one normal measure with Mitchell order 0.

3.1. The first finitely many measurable cardinals.

We begin proving a variation of the classical theorem of Kimchi and Magidor (unpublished) saying that, for each $n < \omega$, the first n measurable cardinals can all be strongly compact and each of them carries any prescribed number of normal measures.

Theorem 3.1. *Assume the* GCH *holds, there are $n<\omega$ supercompact cardinals $\langle\kappa_i\colon i<n\rangle$, there are no measurable cardinals above κ_{n-1}, and each κ_i has a unique normal measure of Mitchell order* 0*. For every $i<n$, let $\tau_i\leq\kappa_i^{++}$ be a cardinal. Then there is a generic extension where* GCH *holds, $\langle\kappa_i\colon i<n\rangle$ are the first n strongly compact cardinals, the first n measurable cardinals, and each κ_i carries exactly τ_i normal measures.*

Proof. We first make the supercompact cardinals $\kappa_0,\ldots,\kappa_{n-1}$ indestructible under sufficiently directed closed forcings which preserve cardinals and GCH. We do this while simultaneously ensuring that for each $i<n$, κ_i carries a unique normal measure of Mitchell order 0. To that end, denote for each $i<n$, $\rho_i:=(\kappa_{i-1})^+$ (if $i=0$, $\rho_0:=0$) and define a finite iterated forcing

$$\mathbb{L}=\langle\mathbb{L}_i,\dot{\mathbb{L}}(\kappa_i,\rho_i)\colon i<n\rangle$$

where $\dot{\mathbb{L}}(\kappa_i,\rho_i)$ is the canonical $\mathbb{L}_i$-name for the ρ_i-directed closed closed forcing which makes the supercompactness of κ_i indestructible under $(\kappa_i)^+$-directed closed forcings, while maintaining the same number of normal measures of Mitchell order 0 on κ_i as in $V^{\mathbb{L}_i}$ (see Notation 2.17).

Claim 3.1.1. $\mathbb{L}$ *preserves cardinals,* GCH*, and the supercompactness of each κ_i. Moreover, each κ_i remains indestructible under $(\kappa_i)^+$-directed-closed forcings and carries a unique normal measure of Mitchell order* 0 *after forcing with* $\mathbb{L}$*.*

Proof of claim. The fact that $\mathbb{L}$ preserves cardinals and the GCH follows from it being a finite iteration of forcings with these properties. Therefore, we focus on the supercompactness and indestructibility of each κ_i.

Let $G\subseteq\mathbb{L}$ be generic over V. Fix $i<n$. We verify that κ_i remains supercompact in $V[G]$. Factor the forcing as $\mathbb{L}=\mathbb{L}_i*\dot{\mathbb{L}}(\kappa_{i+1},\rho_{i+1})*\mathbb{L}\setminus(i+1)$. Since $\mathbb{L}_i\in H(\kappa_i)$, forcing with $\mathbb{L}_i$ does not affect the supercompactness of κ_i, and hence κ_i remains supercompact in $V[G_i]$. In addition, because κ_i has in V a unique normal measure of Mitchell order 0, the same holds in $V[G_i]$. As GCH remains true in $V[G_i]$, Theorem 2.16 applies. It follows that in $V[G_{i+1}]$, the cardinal κ_i is supercompact, carries a unique normal measure of Mitchell order 0, and is indestructible under $(\kappa_i)^+$-directed closed forcings.

Now observe that the tail forcing $\mathbb{L}\setminus(i+1)$ is $(\kappa_i)^+$-directed closed. By indestructibility, κ_i therefore remains supercompact in $V[G]$. Furthermore, no new normal measures on κ_i are added, so in $V[G]$ it carries exactly the same normal measures as in $V[G_{i+1}]$; in particular, it still has a unique normal measure of Mitchell order 0.

It remains to verify that κ_i is indestructible in $V[G]$. Let $\mathbb{Q}\in V[G]$ be a $(\kappa_i)^+$-directed closed poset. Then the two-step forcing $\mathbb{L}\setminus(i+1)*\dot{\mathbb{Q}}$ is itself $(\kappa_i)^+$-directed closed. Since κ_i was already indestructible under such forcings in $V[G_i]$, it follows that forcing with $\mathbb{Q}$ over $V[G]$ preserves the supercompactness of κ_i. This concludes the proof of the claim. □

To simplify the notation, let us assume that V is already a generic extension by $\mathbb{L}$, namely, assume that GCH holds in V, $\kappa_0,\ldots,\kappa_{n-1}$ are supercompact cardinals, each κ_i is indestructible under $(\kappa_i)^+$-directed closed forcings and each κ_i carries a unique normal measure of Mitchell order 0. Consider the product forcing

$$\mathbb{R}=\prod_{i<n}(\dot{\mathbb{P}}^{\tau_i,I_i}*\dot{\mathbb{NR}}(\kappa_{i-1},\kappa_i))$$

where

- I_i is the set of inaccessible cardinals in (κ_{i-1},κ_i) that are limits of strong cardinals (here and below, we let $\kappa_{i-1}=0$ for $i=0$).
- If $\tau_i\leq(\kappa_i)^+$, $\mathbb{P}^{\tau_i,I_i}$ is the spaced splitting forcing from Lemma 2.13.
- If $\tau_i=(\kappa_i)^{++}$, $\mathbb{P}^{\tau_i,I_i}$ is the forcing from Lemma 2.14.
- $\dot{\mathbb{NR}}(\kappa_{i-1},\kappa_i)$ is the nonstationary support iterated forcing which adds a nonreflecting stationary subset to every measurable cardinal in the interval (κ_{i-1},κ_i), as defined in section 2.4.

We argue that in $V^{\mathbb{R}}$, the cardinals $\kappa_0, \dots, \kappa_{n-1}$ are strongly compact, there are no other measurable cardinals, and each κ_i carries exactly τ_i normal measures. This will be proved in the following claim, which concludes the proof of the theorem:

Claim 3.1.2. *Let $i < n-1$. After forcing over V with*

$$\mathbb{R}_{\geq i} = \prod_{j \geq i}(\dot{\mathbb{P}}^{\tau_j, I_j} * \dot{\mathbb{NR}}(\kappa_{j-1}, \kappa_j)),$$

all the cardinals $\kappa_i, \dots, \kappa_{n-1}$ are strongly compact, there are no other measurable cardinals above κ_{i-1}, and each cardinal κ_j for $j \geq i$ carries exactly τ_i normal measures.

Proof of claim. Following the proof of Apter–Cummings [2], we perform a downwards induction. We first establish the claim for $i = n-1$, and the inductive step consists of proving the claim for i, assuming it holds for $i+1$. The theorem follows from the case $i = 0$.

Let us first take care of κ_{n-1} (κ_{n-1} is the top cardinal). Consider the forcing

$$\mathbb{R}_{\geq n-1} = \dot{\mathbb{P}}^{\tau_{n-1}, I_{n-1}} * \dot{\mathbb{NR}}(\kappa_{n-1}, \kappa_n)$$

over V. By Lemmas 2.13 and 2.11, in $V^{\dot{\mathbb{P}}^{\tau_{n-1}, I_{n-1}}}$, GCH holds, κ_{n-1} remains supercompact and carries exactly τ normal measures of Mitchell order 0. Since there are no measurable cardinals above κ_{n-1} in $V^{\dot{\mathbb{P}}^{\tau_{n-1}, I_{n-1}}}$, we can apply Theorem 2.26 to deduce that in $V^{\mathbb{R}_{\geq n-1}}$, κ_{n-1} is a strongly compact cardinal. By Lemma 2.27, κ_{n-1} carries exactly τ_{n-1} normal measures in $V^{\mathbb{R}}$.

Next, we take care of κ_i for $i < n-1$. Consider the forcing

$$\mathbb{R}_{\geq i} = \prod_{j > i}(\dot{\mathbb{P}}^{\tau_j, I_j} * \dot{\mathbb{NR}}(\kappa_{j-1}, \kappa_j)),$$

and factor it to the form

$$\mathbb{R}_{\geq i} = \mathbb{R}_{>i} \times \left(\dot{\mathbb{P}}^{\tau_i, I_i} * \dot{\mathbb{NR}}(\kappa_{i-1}, \kappa_i)\right).$$

The follwing properties hold after forcing over V with $\mathbb{R}_{>i}$:

- κ_i is supercompact, since κ_i is indestructible in in V under $(\kappa_i)^+$-directed closed forcings, and $\mathbb{R}_{>i}$ is such a forcing.
- $\kappa_{i+1}, \dots, \kappa_{n-1}$ are strongly compact, there are no other measurable cardinals above κ_i, and each κ_j (for $j > i$) carries exactly τ_j normal measures. This follows from the induction hypothesis.
- GCH holds, since $\mathbb{R}_{>i}$ preserves GCH as a finite product of forcings which preserve GCH.
- κ_i has a unique normal measure of Mitchell order 0, since this is the case in V, and $\mathbb{R}_{>i}$ is a $(\kappa_i)^{++}$-closed poset.

By Theorem 2.13, all the above bulleted points, except for the last one, are carried on to $V^{\mathbb{R}_{>i} \times \dot{\mathbb{P}}^{\tau_i, I_i}}$. The last bulleted point is now replaced with κ_i carrying exactly τ_i normal measures of Mitchell order 0. Finally, by Theorem 2.24, in $V^{\mathbb{R}_{\geq i}} = V^{\mathbb{R}_{>i} \times \left(\dot{\mathbb{P}}^{\tau_i, I_i} * \dot{\mathbb{NR}}(\kappa_{i-1}, \kappa_i)\right)}$, κ_i becomes strongly compact and the least measurable cardinal above κ_{i-1}. Moreover, By Lemma 2.27, κ_i has exactly τ_i normal measures. The forcing $\mathbb{P}^{\tau_i, I_i} * \dot{\mathbb{NR}}(\kappa_{i-1}, \kappa_i)$ is small relative to the cardinals $\kappa_{i+1}, \dots, \kappa_{n-1}$. Hence their strong compactness, the fact that they are the only measurable cardinals above κ_i, and the number of normal measures they carry all remain unchanged from $V^{\mathbb{R}_{>i}}$. Thus, overall, we obtain that the following properties holds after forcing with $\mathbb{R}_{\geq i}$: $\kappa_i, \kappa_{i+1}, \dots, \kappa_{n-1}$ are strongly compact cardinals, there are no measurable cardinals above κ_{i-1}, and each κ_j (for $j \geq i$) has exactly τ_j normal measures. This concludes the inductive argument. □

This concludes the proof. □

hree-step iteration consisting of: (1) the poset $\mathbb{L}_i$ that makes the supercompactness of κ_i indestructible under $(2^{\kappa_i})^+$-directed-closed forcing; (2) the splitting poset $\mathbb{P}^{\tau_i}$ from page 12 starting the iteration past $(2^{\kappa_i})^+$; (3) the nonstationary support iterations which adds a non-reflecting stationary set without fixed cofinality to every measurable in (κ_{i-1}, κ_i).

3.2. **The first measurable above a supercompact cardinal.** We now turn to the least measurable cardinal above a supercompact cardinal. The Friedman–Magidor technique [14] does not allow one to control the number of normal measures on such a cardinal, since it is carried out over $L[U]$.[33] However, the splitting forcing can be used to control the number of normal measures on the least measurable cardinal above an indestructible supercompact cardinal.

Theorem 3.2. *Assume the* GCH *holds, κ is a supercompact cardinal, and λ is the first measurable cardinal above κ. In addition, suppose that λ has a unique normal measure. Then, for each cardinal $\tau \leq \lambda^{++}$, there is a forcing extension where the following hold:*

(1) κ is supercompact;
(2) λ is measurable;
(3) λ carries exactly τ normal measures.

Proof. Let $\mathbb{L}$ be the Laver preparation rendering the supercompactness of κ indestructible under κ-directed closed forcing (here one may use either Theorem 2.15 or Laver's original construction in [32], since we are not concerned with controlling the measures on κ. If, in addition, one wishes to preserve cardinals and GCH after forcing with $\mathbb{L}$—a feature that may be of independent interest—one may instead apply Theorem 2.16). Let $G \subseteq \mathbb{L}$ be generic over V. By the Lévy–Solovay Theorem (see [34]), λ remains measurable in $V[G]$ and carries a unique normal measure U, necessarily of trivial Mitchell rank.

If $\tau = (\lambda^{++})^{V[G]}$, force with the Easton support iteration adding a single Cohen subset to every inaccessible cardinal in the interval $(\kappa, \lambda]$ (alternatively, apply Lemma 2.14, replacing (κ_0, κ) in the statement of the lemma with (κ, λ)). This blows up the number of normal measures on λ to $(\lambda^{++})^{V[G]}$, while preserving the supercompactness of κ.

If instead $\tau \leq (\lambda^{+})^{V[G]}$, use the splitting forcing $\mathbb{P}^{\tau,I}$, where $I \subseteq \kappa$ is the set of inaccessible cardinals above λ. By Theorem 2.8, if $H \subseteq \mathbb{P}^{\tau,I}$ is generic over $V[G]$, then every normal measure $W \in V[G][H]$ on λ is of the form U^*_η for some $\eta < \tau$. □

3.3. **The first measurable limit of supercompact cardinals.** In this section, we study the number of normal measures on the least cardinal that is a measurable limit of supercompact cardinals. Let us first explain why this cardinal is of interest.

By a well-known theorem of Menas, every measurable cardinal that is a limit of strongly compact cardinals is strongly compact (see [36, Theorem 2.21]). In a recent landmark result, Goldberg proved that, under the Ultrapower Axiom, every strongly compact cardinal that is not supercompact is a measurable limit of supercompact cardinals (see [19, Theorem 8.3.10]). It follows that, under UA, Menas' theorem provides essentially the only way to distinguish between strongly compact and supercompact cardinals.

Assuming the UA or the linearity of the Mitchell order, the least cardinal κ that is a measurable limit of supercompact cardinals is a strongly compact cardinal which carries a unique normal measure (hence, it cannot be supercompact). Indeed, if κ carried two distinct normal measures, then one of them would have nonzero Mitchell order. Consequently, the fact that κ is measurable and a limit of supercompact cardinals would remain true in the corresponding ultrapower. In particular, this property would reflect to a smaller cardinal, contradicting the minimality of κ.

The next theorem shows how to control the number of normal measures on the least measurable limit of supercompact cardinals.

Let us remark that the proof technique below can be applied as well to the least measurable limit of strong cardinals. By a Theorem of Hamkins (see [21, Corollary 2.7]), measurable cardinals which are limits of strong cardinals are tall cardinals. The number of normal measures on tall

[33] More precisely, the Friedman–Magidor construction is performed over an arbitrary model of GCH; however, the first step is a class-forcing iteration coding the universe by a real, thereby transferring it to $L[U, r]$ for some real r. This procedure destroys supercompact cardinals.

cardinals was extensively studied in [3]. Theorem 3.3 provides alternative proofs of some of the results obtained there.

Theorem 3.3. *Assume the* GCH *holds, and suppose that κ is the first measurable limit of supercompact (strong) cardinals. Assume that κ carries a unique normal measure of Mitchell order 0. Let $\tau \leq \kappa^{++}$ be a cardinal. Then, there is a generic extension where κ remains the first measurable limit of supercompact (strong) cardinals, and κ carries exactly τ normal measures.*

Proof. Let κ be the first measurable limit of supercompact (strong) cardinals. Denote by I the set of all inaccessible cardinals below κ that are limits of supercompact (strong) cardinals. Our proof proceeds by forcing with a two-step iteration $\mathbb{Q} = \mathrm{Add}(\omega, 1) * \mathbb{P}$, where $\mathbb{P}$ is an I-spaced nonstationary support iterated forcing ($\mathbb{P}$ will be formally defined below), so that $\mathbb{Q}$ exhibits a gap at ω_1. In particular, forcing with $\mathbb{Q}$ does not create new supercompact, strong or measurable cardinals, by Hamkins' Gap Forcing Theorem (Theorem 2.7). Thus, if $\mathbb{P}$ preserves the fact that κ is a measurable limit of supercompact (strong) cardinals, then it also preserves that κ is the least such cardinal.

Assume first that $\tau \leq \kappa^+$. Let $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \colon \alpha < \kappa \rangle$ be the I-spaced nonstationary support iteration where for each $\alpha \in I$ the $\mathbb{P}_\alpha$-name $\dot{\mathbb{Q}}_\alpha$ is taken as follows:

(1) If $\alpha \in I$ is a limit point of I, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for the atomic forcing $\{\mathbb{1}\} \cup f_\tau(\alpha)$ (here f_τ is the canonical function associated with τ, and $\dot{\mathbb{Q}}_\alpha$ is as in the definition of the Splitting forcing in Section 2.2).

(2) If $\alpha \in I$ is not a limit point of I, let λ_α be the least supercompact (strong) cardinal strictly above α, and let $\dot{\mathbb{Q}}_\alpha$ be the $\mathbb{P}_\alpha$-name for the Laver preparation $\mathbb{L}(\alpha^+, \lambda_\alpha)$ from Notation 2.17 (for the "strong" version of the theorem, $\dot{\mathbb{Q}}_\alpha$ is taken to be $\mathbb{GS}(\alpha^+, \lambda_\alpha)$ from Notation 2.17).

Let $G \subseteq \mathbb{Q}$ be generic over V.

We first argue that κ remains a limit of supercompact (strong) cardinals in $V[G]$. It suffices to prove that, for every cardinal $\alpha \in I$ which is not a limit point of I, λ_α remains supercompact (strong) cardinal in $V[G]$. Indeed, factor

$$\mathbb{Q} = \mathrm{Add}(\omega, 1) * \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha * \mathbb{P} \setminus (\alpha + 1)$$

and note that $\mathrm{Add}(\omega, 1) * \mathbb{P}_\alpha$ preserves the supercompactness (strongness) of λ_α as it is a small forcing relative to λ_α; $\dot{\mathbb{Q}}_\alpha$ preserves the supercompactness (strongness) of λ_α and makes it indestructible under $(\lambda_\alpha)^+$-directed closed forcings; and $\mathbb{P} \setminus (\alpha + 1)$ is $(\lambda_\alpha)^+$-directed closed since $\min(I \setminus (\alpha + 1)) > \lambda_\alpha$. Hence λ_α remains a supercompact (strong) cardinal in $V[G]$.

Next, we argue that κ remains measurable and carries exactly τ normal measures in $V[G]$. By the remarks preceding the theorem, κ carries a unique normal measure U in V. We now aim to invoke Lemma 2.5 to lift the embedding $j_U \colon V \to M_U$ to $V[G]$. Before doing so, we verify that the assumptions of the lemma are satisfied. For Clause ($\beth$), observe that κ is the sole generator of j_U and is therefore bounded by $j_U(\alpha \mapsto \alpha^+)(\kappa) = \kappa^+$, which in turn is smaller than $\min(j_U(I) \setminus (\kappa+1))$.

Since $\kappa \in j_U(I)$, the same argument as in Lemma 2.10 shows that $j_U \colon V \to M_U$ lifts to $V[G]$ in exactly τ ways. More precisely, the embeddings $j^* \colon V[G] \to M^*$ extending j_U are of the form

$$j^*_\eta \colon V[G] \to M_U[G * \{\langle \kappa, \eta \rangle\} * (j_U[G] \setminus \kappa + 1)]$$

for some $\eta < \tau$, and each j^*_η induces a distinct normal measure U^*_η on κ in $V[G]$.

On the other hand, suppose that $W \in V[G]$ is an arbitrary normal measure in κ. By the Gap Forcing Theorem, W lifts a normal measure $U \in V$. Since κ is a measurable limit of supercompact (strong) cardinals in V, $I \in U$. In particular $I \in W$ and thus $\kappa \in j_W(I)$. Arguing exactly as in the last paragraph of the proof of Theorem 2.8 we conclude that $W = U^*_\eta$ for some $\eta < \tau$. Thereby, κ carries exactly τ normal measures in $V[G]$.

This concludes the proof of the theorem for $\tau \leq \kappa^+$. If $\tau = \kappa^{++}$, modify $\mathbb{P}$ to an iterated forcing of length $\kappa + 1$, such that, for every $\alpha \in I \cup \{\kappa\}$ which is a limit point of I, $\dot{\mathbb{Q}}_\alpha$ is forced to be $(\mathrm{Add}(\alpha,1))^{V^{\mathbb{P}_\alpha}}$. Standard arguments show that κ remains measurable and carries κ^{++} normal measures in $V[G]$. The same argument as above shows that κ is a limit of supercompact (strong) cardinals in $V[G]$. □

Remark 3.4. In Theorem 3.3 we can obtain GCH in the generic extension by using the version of the indestructibility theorem under forcings which preserve GCH (2.16).

3.4. **A generalization of the Goldberg–Woodin Theorem.** One of the main results in [17] is the proof that, assuming the consistency of the Ultrapower Axiom[34] with a supercompact cardinal, it is consistent that the least measurable cardinal is strongly compact and has a unique normal measure. The proof was performed over a model of the Ultrapower Axiom in which there is a supercompact cardinal, and there are no measurable cardinals above it (this configuration could be achieved by cutting the universe at the least measurable cardinal above the least supercompact cardinal). The first author asked whether the anti-large cardinal component of the proof can be omitted. In an unpublished work prior to [17], Goldberg and Woodin proved the same result without relying on any anti-large cardinal assumption, starting from a model of UA with a measurable limit of supercompact cardinals. Building on the Goldberg–Woodin approach, we prove that under the same assumption, the first measurable cardinal can be strongly compact and have any prescribed number of normal measures, without appealing to anti-large cardinal assumptions in the proof.

Theorem 3.5. *Assume the* GCH *holds, κ is a measurable limit of supercompact cardinals, $\tau \leq \kappa^{++}$, and κ has a unique normal measure of Mitchell order 0.*[35] *Then there is a generic extension where κ is the least measurable cardinal, the least strongly compact cardinal, and κ carries exactly τ normal measures.*

Proof. By Theorem 3.3 and Remark 3.4, we can assume that GCH holds, κ is the least measurable limit of supercomapct cardinals, and κ has τ normal measures. Next, perform a Magidor iteration $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \colon \alpha < \kappa\rangle$ that adds a Prikry sequence to any measurable cardinal below κ (including the supercomapct cardinals below κ). In the resulting model, κ is the least measurable cardinal and a strongly compact cardinal (see [35] or [16] for more details on the Magidor iteration of Prikry forcings and the preservation of the strongly compact cardinal). Finally, by [27, Theorem 0.1], κ has exactly τ normal measures in the generic extension. □

4. A comment on Woodin's HOD hypothesis

Woodin's HOD hypothesis asserts that there is a proper class of regular cardinals that are not ω-strongly measurable in HOD. It is motivated by Woodin's landmark HOD Dichotomy Theorem:

Theorem 4.1 (Woodin [44])**.** *Let κ be an extendible cardinal. Then exactly one of the following holds:*

(1) HOD *is a weak extender model for the supercompactness of κ.*
(2) Every regular cardinal $\geq \kappa$ is ω-strongly measurable in HOD*.*

We refer to [44] for the relevant definitions.

In [20], Goldberg-Poveda proved the consistency of Woodin's HOD hypothesis with the first supercompact cardinal κ being strongly compact in HOD yet not 2^κ-supercompact in HOD. Therefore, if κ is merely supercompact (and not HOD-supercompact, as required by Woodin), HOD may fail to be a weak extender model for the supercompactness of the first supercompact cardinal κ – even under the HOD hypothesis.

[34]As mentioned above, weaker consequences of UA suffice for the proof.

[35]If κ is the least measurable limit of supercompact cardinals and the Mitchell order is linear, then, by its minimality, κ carries a unique normal measure of order 0. Thus, under UA, the least measurable limit of supercompact cardinals fulfills the assumptions in the theorem.

In this brief section we further strengthen this disagreement between V and HOD by producing the consistency of the HOD hypothesis with the first supercompact cardinal carrying a unique normal measure in HOD.

We will need the natural nonstationary support variation of the family of self-coding iterations considered in [20]:

Definition 4.2. Let θ be an ordinal. A nonstationary support *self-coding iteration* is a sequence

$$\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha, \dot{A}_\alpha \colon \alpha < \theta \rangle$$

such that the following hold for every Mahlo cardinal α, where t_α denotes the transitive closure of $\mathbb{P}_\alpha$ and G_α denotes the canonical $\mathbb{P}_\alpha$-name for a generic set:

(1) $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a poset that is forced by the trivial condition to be $|\mathbb{P}_\beta|$-closed, for all $\beta < \alpha$.
(2) $\dot{A}_\alpha$ is a $\mathbb{P}_\alpha$-name forced by the trivial condition to be a binary relation on $|t_\alpha|^V$ whose transitive collapse is $t_\alpha \cup \{\dot{G}_\alpha\}$.
(3) $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha * \mathrm{Code}(\dot{A}_\alpha, \gamma)$ where γ is the first inaccessible above $|\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha|$ and $\mathrm{Code}(\dot{A}_\alpha, \gamma)$ is the coding poset from [20, §2].

As usual, when defining $\mathbb{P}$ at limit stages α, the iteration $\mathbb{P}_\alpha$ is constructed as the nonstationary support limit if α is inaccessible or as the inverse limit otherwise.

The following fact was proved in [20, Lemma 2.21] for self-coding iterations with Easton support. The proof for self-coding iterations with nonstationary support is similar:

Fact 4.3. *Suppose that θ is a Mahlo cardinal and let $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha, \dot{A}_\alpha, \dot{\mathbb{Q}}_\alpha \colon \alpha < \theta \rangle$ be a a self-coding iteration. If $G \subseteq \mathbb{P}_\theta$ is generic over V then G is definable over the structure $(V[G]_\theta, V_\theta, \vec{\mathbb{P}})$ without parameters. In particular, if $V \subseteq \mathrm{HOD}^{V[G]}$ and $\vec{\mathbb{P}}$ is definable over V_θ then*

$$\mathrm{HOD}^{V[G]} = V[G].$$

Theorem 4.4. *Assume* GCH *and* $V = \mathrm{gHOD}$ *both hold*[36]*, κ is a supercompact cardinal, there are no inaccessible cardinals above κ and there exists a unique normal measure on κ with Mitchell order* 0*. Then, the following configuration is consistent:*

(1) The HOD *hypothesis holds.*
(2) κ is supercompact.
(3) In HOD*, κ is strongly compact and it carries exactly one normal measure.*

Proof. Fix $\ell \colon \kappa \to V_\kappa$ a Laver function. We shall denote by I the set of measurable cardinals $\alpha < \kappa$ that are closure points of ℓ, namely

$$I = \{\alpha < \kappa \colon \alpha \text{ is measurable } \wedge \forall \beta < \alpha\, (\ell(\beta) \in \mathrm{ORD} \to \ell(\beta) < \alpha)\}.$$

Let $\mathbb{P}_\kappa$ the I-spaced, nonstationary support self-coding iteration of length κ such that $\dot{\mathbb{Q}}_\alpha$ is forced to be $\mathrm{Add}(\alpha, 1)$ at every $\alpha \in I$.

By Fact 4.3, for every $G \subseteq \mathbb{P}_\kappa$ generic over V we have that

$$V[G] = \mathrm{HOD}^{V[G]}.$$

Our intended generic extension is obtained by forcing with $\mathbb{P}_\kappa * \dot{\mathrm{Add}}(\kappa, 1)$. Let $G * H \subseteq \mathbb{P}_\kappa * \mathrm{Add}(\kappa, 1)$ a generic over V.

$$\mathrm{HOD}^{V[G][H]} = V[G]$$

by homogeneity of $\mathrm{Add}(\kappa, 1)^{V[G]}$ and because this forcing does not disturb the coding into the power-set function pattern (see [20] for details)

Claim 4.4.1. *In $\mathrm{HOD}^{V[G][H]}$, κ carries exactly one normal measure.*

[36]Recall that $\mathrm{gHOD} := \{x \in V \colon \text{ for every poset } \mathbb{P} \in V,\ (\Vdash_{\mathbb{P}} \check{x} \in \mathrm{HOD})\}$. See [15].

Proof. By our previous observations, it suffices to argue the claim in $V[G]$. Let U be the unique normal measure of trivial Mitchell rank in V. By the argument of Claim 2.15.1, U generates in $V[G]$ a normal measure U^*.

Suppose for the sake of a contradiction W is a normal measure on κ in $V[G]$ different from U^*. Then, by the Gap Forcing Theorem, $\bar{W} = W \cap V$ belongs to V and the ultrapower of $V[G]$ under W, M_W, is of the form $M_W = M[j_W(G)]$ where $M = M_W \cap V$.

Crucially, $M = M_{\bar{W}}$. To show this, let $k\colon M_{\bar{W}} \to M$ be the map defined by

$$k([f]_{\bar{W}}) := [f]_W.$$

It is not hard to verify that k is elementary and that $j_W \restriction V = k \circ j_{\bar{W}}$. To complete the proof of the equality it is enough to argue that k is the identity function. Assume otherwise, and denote $\mu = \mathrm{crit}(k)$. Since $\bar{W}, W$ are normal measures, $\mu \geq (\kappa^{++})^{M_{\bar{W}}}$. Write $\mu = [f]_W$ for some $f\colon \kappa \to \mathrm{Ord}$ in $V[G]$. By Lemma 2.4, there exists $F \in V$ and a club $C \subseteq \kappa$ in V, such that, for every $\alpha \in C$, $f(\alpha) \in F(\alpha)$ and $|F(\alpha)| \leq |\mathbb{P}_{\alpha+1}|^V = \alpha^+$. In particular, by normality of W, $\kappa \in j_W(C)$, so

$$\mu = j_W(f)(\kappa) \in j_W(F)(\kappa) = k(j_{\bar{W}}(F))(\kappa) = k(j_{\bar{W}}(F)(\kappa)) = k[j_{\bar{W}}(F)(\kappa)]$$

where in the second-to-last equality we used the fact that $\mathrm{crit}(k) > \kappa$, and in the last equality we used the fact that $|j_{\bar{W}}(F)(\kappa)| \leq \kappa^+ < \mathrm{crit}(k)$. Overall, we proved that $\mu \in \mathrm{Im}(k)$, contradicting the fact that $\mu = \mathrm{crit}(k)$.

Note that since we are assuming that $W \neq U^*$ it cannot be the case that $\bar{W} = U$. Therefore, by our assumption in V, the measure $\bar{W}$ must have Mitchell-order ≥ 1 in V. Thus we have an elementary embedding $j_W\colon V[G] \to M_{\bar{W}}[j_W(G)]$ where $M_{\bar{W}} \models$ "κ is measurable". In particular, this means that $j_W(\mathbb{P}_\kappa)$ has added a Cohen subset to $M_{\bar{W}}[j_W(G) \cap j_W(\mathbb{P}_\kappa)_\kappa] = M_{\bar{W}}[G]$ at stage κ. Next, we argue that this Cohen subset of κ must be $V[G]$-generic which yields the sought contradiction with $W \neq U^*$.

First, $\mathrm{Add}(\kappa, 1)^{V[G]} = \mathrm{Add}(\kappa, 1)^{M_{\bar{W}}[G]}$ because $M_{\bar{W}}[G]$ is an inner model of $V[G]$ closed under κ-sequences of its elements by Claim 2.5.1. Second, any maximal antichain $A \in V[G]$ in $(\mathrm{Add}(\kappa, 1)^{V[G]}$ is a subset of $V[G]_\kappa$, so again by Claim 2.5.1, $A \in M_{\bar{W}}[G]$.[37] Thus, any $M_{\bar{W}}[G]$-generic for $\mathrm{Add}(\kappa, 1)^{V[G]}$ is generic over $V[G]$. □

An argument simpler than the one given in the proof of Theorem 2.24 shows that κ remains strongly compact in $V[G]$.[38]

Claim 4.4.2. *In $V[G]$, κ is strongly compact.*

Proof. Let $\lambda > \kappa$ be regular, U_0 the unique normal measure on κ with trivial Mitchell rank and $F \in M_{U_0}$ a normal, fine ultrafilter on $\mathcal{P}_{j_{U_0}(\kappa)}(j_{U_0}(\lambda))^{M_U}$ such that $j_F(j_{U_0}(\ell))(j_{U_0}(\kappa)) = j_{U_0}(\lambda)$. We will lift the embedding

$$j = j_F \circ j_{U_0}\colon V \to M \simeq \mathrm{Ult}(M_{U_0}, F).$$

First, since $\mathbb{P}_\kappa$ has nonstationary support and U_0 has a trivial Mitchell order, j_{U_0} lifts to

$$j_{U_0}\colon V[G] \to M_{U_0}[G * j_{U_0}[G] \setminus \kappa + 1] = M_{U_0}[j_{U_0}[G]].$$

Second, by elementarity and the definition of $\mathbb{P}_\kappa$, $j(\mathbb{P}_\kappa)$ factorises as

$$j_{U_0}(\mathbb{P}_\kappa) * \dot{\mathrm{A}}\mathrm{dd}(j_{U_0}(\kappa), 1) * \dot{\mathbb{Q}},$$

where the latter poset is forced to be $j_{U_0}(\lambda)^{+M_U[j_{U_0}(G)]}$-directed closed.

[37]Indeed, A can be coded as a κ-sequence $\vec{X} = \langle A \cap (V_\beta)^{V[G]} : \beta < \kappa \rangle$, and, since $A \cap (V_\beta)^{V[G]} \in M_{\bar{W}}[G]$ for every $\beta < \kappa$, $\vec{X} \in M_{\bar{W}}[G]$.

[38]The simplicity comes from the fact that this time our iteration uses Cohen forcing in place of $\mathbb{NR}(\alpha)$; that is, now the forcing at each stage α is α-directed closed, while before, the forcings $\mathbb{NR}(\alpha)$ only had arbitrarily high dense directed closed subsets.

Let $g \in V[G]$ be an $M[j_{U_0}(G)]$-generic filter for $\mathrm{Add}(j_{U_0}(\kappa),1)^{M[j_{U_0}[G]]}$. Constructing such g in $V[G]$ is possible, since $(\mathrm{Add}(j_{U_0}(\kappa),1))^{M[j_{U_0}[G]]}$ is a κ^+-closed forcing whose dense open sets in $M[j_{U_0}[G]]$ can be enumerated in $V[G]$ in order-type κ^+.

Since $\dot{\mathbb{Q}}$ is sufficiently directed closed, the set

$$\{j(p)\setminus j_{U_0}(\kappa)\colon p\in G\}=\{\{\mathbb{1}\}^\frown j(p)\setminus j_{U_0}(\kappa)+1\colon p\in G\}$$

admits a lower bound $q \in j(\mathbb{P})\setminus j_{U_0}(\kappa)$ with $q(j_{U_0}(\kappa))=\mathbb{1}$. Using q as a master condition, we can build $L \in M_U[j_{U_0}(G)*g]$ a $j(\mathbb{P})\setminus(j_{U_0}(\kappa)+1)$-generic over $M[j_{U_0}[G]*g]$ such that $q \in g*L$. Therefore, $j\colon V\to M$ lifts to

$$j\colon V[G]\to M[j_{U_0}(G)*g*L]$$

that is a λ-strongly compact embedding. □

Claim 4.4.3. *In $V[G]$, κ is not 2^κ-supercompact.*

Proof. Suppose otherwise and let $j\colon V[G]\to M$ be a witnessing embedding in $V[G]$. Then κ is measurable in M. By the Gap Forcing Theorem 2.7, $M=\bar{M}[H]$ for some subclass $\bar{M}\subseteq V$ and a $j(\mathbb{P}_\kappa)$-generic H over $\bar{M}$.

By the Gap Forcing Theorem 2.7 forcing with $j(\mathbb{P}_\kappa)$ does not create new measurable cardinals. As a result, κ is measurable in $\bar{M}$ as well. In addition, κ is a closure point of $j(\ell)$ so, all in all, $\kappa\in j(I)$. This means that at stage κ the iteration $j(\mathbb{P}_\kappa)$ has opted to force with $\mathrm{Add}(\kappa,1)$. Using exactly the same argument given in Claim 4.4.1 one demonstrates that this Cohen is in fact generic over $V[G]$, which produces the sought contradiction.

□

After forcing with $H\subseteq \mathrm{Add}(\kappa,1)_{V[G]}$ the supercompactness of κ is restored: Indeed, in $M[G]$ the forcing used at stage κ is of the form $\mathrm{Add}(\kappa,1)_{V[G]}*\mathrm{Code}(A_\kappa,\gamma)_{M[G]}$ where γ is the first $M[G]$-inaccessible above κ. Since we were assuming that no inaccessible cardinal greater than κ exists it follows that the first $M[G]$-inaccessible greater than κ is above λ. As a result, standard arguments allow us to define a $M[G]$-generic filter for $\mathrm{Code}(A_\kappa,\gamma)_{M[G]}$ in $V[G*H]$. These considerations together with standard arguments of Silver (see [13, Theorem 12.6]) finally show that κ is supercompact in $V[G*H]$. □

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